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Tracking Control Design for Non-Standard Nonlinear Singularly Perturbed Systems

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Abstract— **Tracking control laws for a general class of nonlinear singularly perturbed systems are developed. No assumptions concerning the nonlinearity of the system is made. The effect of the different speeds of controllers and nonlinear actuator dynamics is studied and asymptotic stabilization is shown using Lyapunov methods. Design procedure and performance of the proposed technique is evaluated against composite control method. Results indicate that the proposed technique applies both to standard and non-standard forms of singularly perturbed systems.**

I. INTRODUCTION

Analysis and control of singularly perturbed systems has received considerable attention in literature [1]. The common approach is to design two separate controllers for each of the two lower-order subsystems and then apply their composite or sum to the full-order system. The composite control technique[2] guarantees asymptotic stability for standard singularly perturbed systems or for systems wherein the algebraic problem has a unique root for the fast variables in the region of interest. In literature this assumption is satisfied by either assuming that a unique root for the fast states exists [3] or assuming that the system dynamics is nonlinear only in the slow states [4]. However, this root is a set of fixed points of the fast dynamics expressed as a smooth function of the slow variables and the control inputs, and hence is not always unique nor guaranteed to exist. Consequently one is required to choose an isolated manifold in order to design a stabilizing control structure for the slow subsystem. This not only requires substantial system knowledge but also restricts the results to a local domain. Furthermore, analytical determination of this manifold is restricted by the nonlinearity of the system. In such cases, it has been shown that only ultimate boundedness of the signals maybe concluded [5].

This paper proposes an alternate approach for control design of non-standard forms of singularly perturbed systems. The proposed approach avoids analytical computation of the manifold by considering it as an additional control variable. This idea is motivated by singularly perturbed systems such as aircraft wherein the fast states appear linearly in the slow dynamics. Reference [6] successfully designed nonlinear flight trajectories using angular rates as control variables, although, the effect of control variables on the slow variables was neglected. More recently ultimate uniformly bounded results were concluded [7] using similar ideas while assuming that the set of nonlinear algebraic equations can be solved for the control variables and the fast controller was designed using gain-scheduling.

This paper makes three major contributions. First, the approach developed here employs the reduced-order technique without imposing any assumptions about the solutions of the transcendental equations or the effect of the control variables. By computing the slow manifold upon which the fast states must be restricted for asymptotic tracking and ensuring that this manifold is the equilibrium of the system uniformly, control objective is accomplished. Second, controllers with different speeds are addressed in comparison to composite control technique that requires all control variables to be sufficiently fast. Third, the control laws are computed using Lyapunov-based designs that are able to capture the nonlinear behaviour that is lost in the linearization of the system. Owing to this, the global or local nature of results are relaxed from the complexities of analytic construction of the manifold and are entirely a consequence of the choice of underlying controllers for the reduced-order models. Additionally, the control laws developed in this paper are independent of the singular perturbation parameter and an upper bound for the scalar perturbation parameter is derived as a sufficiency condition for asymptotic stability.

The paper is organized as follows. Section II mathematically formulates the control problem and briefly reviews the necessary concepts from geometric singular perturbation theory. Control laws and the main results of the paper are detailed in Section III. Section IV studies several numerical examples and qualitatively analyses the performance and design procedure of the proposed technique. Conclusions are discussed in Section V.

II. PROBLEM DESCRIPTION AND MODEL REDUCTION

A. System Description

The class of nonlinear singularly perturbed dynamical systems addressed in this paper are

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \delta); \quad \mathbf{x} \in \mathbb{R}^m, \delta \in \mathbb{R}^p && \text{(1a)}\\
\dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}); \quad \delta_{\mathbf{1}} \in \mathbb{R}^l, \mathbf{u_1} \in \mathbb{R}^l && \text{(1b)}\\
\epsilon\dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \delta, \epsilon); \quad \mathbf{z} \in \mathbb{R}^n && \text{(1c)}\\
\epsilon\dot{\delta}_{\mathbf{2}} &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}); \quad \delta_{\mathbf{2}} \in \mathbb{R}^{p-l}, \mathbf{u_2} \in \mathbb{R}^{p-l} && \text{(1d)}
\end{aligned}$$

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where $\mathbf{x}$ is the vector of slow variables, $\mathbf{z}$ is the vector of fast variables, $\delta = [\delta_{\mathbf{1}}, \delta_{\mathbf{2}}]^T$ is the vector of actuator commands input to the system, $\mathbf{u} = [\mathbf{u_1}, \mathbf{u_2}]^T \in \mathbb{R}^p$ is the input vector that is to be computed and $\epsilon \in \mathbb{R}$ is the singular perturbation parameter that satisfies $0 < \epsilon << 1$ and is unknown. All the vector fields are assumed to be sufficiently smooth and $p \geq m$. The control objective is to drive the slow state so as to track sufficiently smooth, bounded, time-varying trajectories or, $\mathbf{x}(t) \to \mathbf{x}_r(t)$ as $t \to \infty$.

The controls have been separated into vectors δ_1 and δ_2 to consider the different speeds of the control variables, with δ_1 representing the actuators with slow dynamics and δ_2 representing actuators with relatively fast actuator dynamics. The vector fields $\mathbf{f}_{\delta_1}(.)$ and $\mathbf{f}_{\delta_2}(.)$ represent their actuator dynamics respectively. The model given in (1) represents the special case of two time-scale dynamical systems. The design procedure developed here also applies to multiple time-scale systems of the following form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \delta) \\ \epsilon_1 \dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) \\ \epsilon_2 \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \delta, \epsilon_2) \\ \epsilon_3 \dot{\delta}_{\mathbf{2}} &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) \end{aligned}$$

where ϵ_1, ϵ_2 and ϵ_3 are singular perturbation parameters of different orders that satisfy $\epsilon_3 < \epsilon_2 < \epsilon_1$.

B. Reduced-Order Models

The system considered in (1) is labeled the *Slow System* and the independent variable t is called the *slow time-scale*. This system is equivalently written as

$$\begin{aligned} \mathbf{x}' &= \epsilon \mathbf{f}(\mathbf{x}, \mathbf{z}, \delta) & \text{(2a)} \\ \delta_{\mathbf{1}}' &= \epsilon \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) & \text{(2b)} \\ \mathbf{z}' &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \delta, \epsilon) & \text{(2c)} \\ \delta_{\mathbf{2}}' &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) & \text{(2d)} \end{aligned}$$

where $'$ represents derivative with respect to $\tau = \frac{t - t_0}{\epsilon}$ and t_0 is the initial time. Equation (2) are labeled the *Fast System* and the independent variable τ is called the *fast time-scale*. Geometric singular perturbation theory[8] examines the behaviour of these singularly perturbed systems by studying the geometric constructs of the reduced-order models which are obtained by substituting $\epsilon = 0$ in (1) and (2). This results in:

Reduced Slow Subsystem:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}, \delta) & \text{(3a)} \\ \dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) & \text{(3b)} \\ \mathbf{0} &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \delta, 0) & \text{(3c)} \\ \mathbf{0} &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) & \text{(3d)} \end{aligned}$$

Reduced Fast Subsystem:

$$\begin{aligned} \mathbf{x}' &= \mathbf{0}; \quad \delta_{\mathbf{1}}' = \mathbf{0} & \text{(4a)} \\ \mathbf{z}' &= \mathbf{g}(\mathbf{x}, \mathbf{z}, \delta, 0) & \text{(4b)} \\ \delta_{\mathbf{2}}' &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) & \text{(4c)} \end{aligned}$$

The dynamics of the resulting reduced slow subsystem are restricted to $m + l$ dimensions, constrained to lie upon an $n + p - l$ dimensional smooth manifold defined by the set of points $(\mathbf{x}, \mathbf{z}, \delta) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ that satisfy the algebraic equations (3c),(3d):

$$\mathcal{M}_0 : \mathbf{z} = \mathbf{z}(\mathbf{x}, \delta_{\mathbf{1}}, \delta_{\mathbf{2}}); \delta_{\mathbf{2}} = \delta_{\mathbf{2}}(\mathbf{u}_2) \quad (5)$$

This set of points is identically the fixed points of the reduced fast subsystem (4b)-(4c). Thus the manifold $\mathcal{M}_0$ is invariant [9]. Furthermore, the flow on this manifold is described by the differential equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{z}(\mathbf{x}, \delta_{\mathbf{1}}, \delta_{\mathbf{2}}), \delta_{\mathbf{1}}, \delta_{\mathbf{2}}(\mathbf{u}_2)) & \text{(6a)} \\ \dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) & \text{(6b)} \end{aligned}$$

if the reduced fast subsystem is stable about the manifold $\mathcal{M}_0$. If the dynamics of (6) are locally asymptotically stable about the manifold, then it can be concluded that the complete system (1) is also locally asymptotically stable [9].

III. CONTROL FORMULATION AND STABILITY ANALYSIS

Stability properties of the slow system depend upon the identification of the manifold $\mathcal{M}_0$. In general, the nonlinear set of algebraic equations (3c),(3d) possess multiple roots and the manifold $\mathcal{M}_0$ may take any of these values; hence it is not unique. One approach to ensure uniqueness is to consider the fast state as another control variable. These ideas are mathematically formulated and analyzed in this section.

A. Control Design

The first step is to transform the problem into a non-autonomous stabilization control problem. Define the tracking error signal as $\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_r(t)$ and express the slow system as

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{F}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta) & \text{(7a)} \\ \dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) & \text{(7b)} \\ \epsilon \dot{\mathbf{z}} &= \mathbf{G}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \delta, \epsilon) & \text{(7c)} \\ \epsilon \dot{\delta}_{\mathbf{2}} &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) & \text{(7d)} \end{aligned}$$

where $\mathbf{F}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta) = \mathbf{f}(\mathbf{e} + \mathbf{x}_r, \mathbf{z}, \delta) - \dot{\mathbf{x}}_r$ and $\mathbf{G}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \delta) = \mathbf{g}(\mathbf{e} + \mathbf{x}_r, \mathbf{z}, \delta, \epsilon)$ are Lipschitz on a domain of the state-space. Using the procedure described in Section II, the reduced slow subsystem for set of equations in (7) is obtained as

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{F}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta) & \text{(8a)} \\ \dot{\delta}_{\mathbf{1}} &= \mathbf{f}_{\delta_1}(\delta_{\mathbf{1}}, \mathbf{u_1}) & \text{(8b)} \\ \mathbf{0} &= \mathbf{G}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \delta, 0) & \text{(8c)} \\ \mathbf{0} &= \mathbf{f}_{\delta_2}(\delta_{\mathbf{2}}, \mathbf{u_2}) & \text{(8d)} \end{aligned}$$

In order to ensure $\mathbf{e} = \mathbf{0}$ is an asymptotically stable equilibrium of the reduced slow system (8) define a positive-definite and decrescent Lyapunov function that satisfies

Condition 1. $V(t, \mathbf{e}) : [0, \infty) \times D_{\mathbf{x}} \to \mathbb{R}$ is continuously differentiable and $D_{\mathbf{x}} \subset \mathbb{R}^m$ contains the origin, such that

$$0 < \psi_1(||\mathbf{e}||) \leq V(t, \mathbf{e}) \leq \psi_2(||\mathbf{e}||)$$

for some **class** $\mathcal{K}$ functions $\psi_1(.)$ and $\psi_2(.)$.

Let $\delta_{\mathbf{2r}}$ represent the manifold of the equation (8d) that is defined shortly. Design a manifold $\mathbf{z} = \mathbf{z_r}(\mathbf{e}, \mathbf{x_r}, \dot{\mathbf{x}}_\mathbf{r}, \delta_{\mathbf{2r}})$ and control $\delta_\mathbf{1} = \delta_{\mathbf{1r}}(\mathbf{e}, \mathbf{x_r}, \dot{\mathbf{x}}_\mathbf{r}, \delta_{\mathbf{2r}})$ such that the slow state error system (8a) satisfies

Condition 2.

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) \leq -\alpha_1 \psi_3^2(\mathbf{e}), \quad \alpha_1 > 0$$

where $\psi_3(.)$ is a continuous positive-definite scalar function that satisfies $\psi_3(\mathbf{0}) = 0$.

The next step is to design control $\mathbf{u_1}$ that ensures the actuator state asymptotically approaches $\delta_{\mathbf{1r}}$. Define the error in actuator state as $\mathbf{e}_{\delta_1} := \delta_\mathbf{1} - \delta_{\mathbf{1r}}$ and rewrite the reduced slow error subsystem (8a),(8b) as

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) + \\ &\quad \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}) - \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) \end{aligned} \tag{9a}$$

$$\dot{\mathbf{e}}_{\delta_1} = \mathbf{f}_{\delta_1}(\delta_\mathbf{1}, \mathbf{u_1}) - \dot{\delta}_{\mathbf{1r}} \tag{9b}$$

where $\dot{\delta}_{\mathbf{1r}}$, the derivative of the manifold is given as

$$\begin{aligned} \dot{\delta}_{\mathbf{1r}} &= \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{e}}\dot{\mathbf{e}} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{x_r}}\dot{\mathbf{x}}_\mathbf{r} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \dot{\mathbf{x}}_\mathbf{r}}\ddot{\mathbf{x}}_\mathbf{r} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \delta_{\mathbf{2r}}}\dot{\delta}_{\mathbf{2r}} \\ &= \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}) + \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{x_r}}\dot{\mathbf{x}}_\mathbf{r} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \dot{\mathbf{x}}_\mathbf{r}}\ddot{\mathbf{x}}_\mathbf{r} \end{aligned} \tag{10}$$

using (9a) and the fact that $\delta_{\mathbf{2r}}$ is a fixed point of the reduced slow subsystem as it satisfies equation (8d). Conditions $1-2$ ensure that the slow error is asymptotically stabilized by the slow actuator variable $\delta_{\mathbf{1r}}$. In order to ensure the system remains asymptotically stable when $\mathbf{e}_{\delta_1} \neq 0$, define a combined positive-definite decrescent Lyapunov function for equations (9a),(9b) such that $V_s(t, \mathbf{e}, \mathbf{e}_{\delta_1}) : [0, \infty) \times D_\mathbf{x} \times D_{\delta_1} \to \mathbb{R}$ is continuously differentiable and $D_{\delta_1} \subset \mathbb{R}^l$ contains the origin

$$V_s(t, \mathbf{e}, \mathbf{e}_{\delta_1}) = V(t, \mathbf{e}) + \frac{1}{2}{\mathbf{e}_{\delta_1}}^T \mathbf{e}_{\delta_1} \tag{11}$$

and design $\mathbf{u_1}$ such that the closed-loop reduced slow system (9) satisfies

Condition 3.

$$\frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial \mathbf{e}}\dot{\mathbf{e}} + \frac{\partial V_s}{\partial \mathbf{e}_{\delta_1}}\dot{\mathbf{e}}_{\delta_1} \leq -\alpha_1 \psi_3^2(\mathbf{e}) - \alpha_2 \psi_4^2(\mathbf{e}_{\delta_1}), \quad \alpha_2 > 0$$

where $\psi_4(.)$ is a continuous positive-definite scalar function that satisfies $\psi_4(\mathbf{0}) = 0$.

Conditions $1-3$ complete the design of control for the reduced slow subsystem. Notice that the manifold $\mathbf{z_r}(\mathbf{e}, \mathbf{x_r}, \dot{\mathbf{x}}_\mathbf{r}, \delta_{\mathbf{2r}})$ computed in the above control design is a function of the manifold $\delta_{\mathbf{2r}}$ which is unknown. From the discussion detailed in Section II, it is known that this manifold is a fixed point of the reduced fast subsystem,

$$\mathbf{e}' = \mathbf{0}; \quad {\delta_1}' = \mathbf{0} \tag{12a}$$

$$\mathbf{z}' = \mathbf{G}(\mathbf{e}, \mathbf{z}, \mathbf{x}_r, \delta, 0) \tag{12b}$$

$${\delta_\mathbf{2}}' = \mathbf{f}_{\delta_\mathbf{2}}(\delta_\mathbf{2}, \mathbf{u_2}) \tag{12c}$$

The complete system will have the properties of the reduced slow subsystem if the fast state asymptotically stabilizes about $\mathbf{z_r}$. This condition is enforced by designing the manifold $\delta_{\mathbf{2r}}$. Define the error in the fast state vector $\mathbf{e_z} := \mathbf{z} - \mathbf{z_r}$ and rewrite (12b) as

$${\mathbf{e_z}}' = \mathbf{G}(\mathbf{e}, \mathbf{e_z}, \mathbf{x}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}, 0) \tag{13}$$

while noting that ${\mathbf{z_r}}' = \epsilon \dot{\mathbf{z}}_\mathbf{r} = \mathbf{0}$ for the reduced fast subsystem. Define a positive-definite and decrescent Lyapunov function that satisfies

Condition 4. $W(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e_z}) : [0, \infty) \times D_\mathbf{x} \times D_{\delta_1} \times D_\mathbf{z} \to \mathbb{R}$ is continuously differentiable and $D_\mathbf{z} \subset \mathbb{R}^n$ contains the origin, such that

$$0 < \phi_1(||\mathbf{e_z}||) \leq W(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e_z}) \leq \phi_2(||\mathbf{e_z}||)$$

for some **class** $\mathcal{K}$ functions $\phi_1(.)$ and $\phi_2(.)$.
and design $\delta_{\mathbf{2r}}$ such that the closed-loop reduced fast system (13) satisfies

Condition 5.

$$\frac{\partial W}{\partial \mathbf{e_z}}\mathbf{G}(\mathbf{e}, \mathbf{e_z}, \mathbf{x}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}, 0) \leq -\alpha_3 \phi_3^2(\mathbf{e_z}), \quad \alpha_3 > 0$$

where $\phi_3(.)$ is a continuous positive-definite scalar function that satisfies $\phi_3(\mathbf{0}) = 0$.

Thus, the last step in the design procedure is to enforce that the fast actuators asymptotically stabilize about $\delta_{\mathbf{2r}}$ and the closed-loop reduced fast subsystem is uniformly stable. Define the error in the fast actuator states $\mathbf{e}_{\delta_2} := \delta_\mathbf{2} - \delta_{\mathbf{2r}}$ and rewrite the closed-loop reduced fast subsystem in the error coordinates

$$\begin{aligned} {\mathbf{e_z}}' &= \mathbf{G}(\mathbf{e}, \mathbf{e_z}, \mathbf{x}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}, 0) + \\ &\quad \mathbf{G}(\mathbf{e}, \mathbf{e_z}, \mathbf{x}_r, \delta_\mathbf{1}, \delta_\mathbf{2}, 0) - \mathbf{G}(\mathbf{e}, \mathbf{e_z}, \mathbf{x}_r, \delta_\mathbf{1}, \delta_{\mathbf{2r}}, 0) \end{aligned} \tag{14a}$$

$${\mathbf{e}_{\delta_2}}' = \mathbf{f}_{\delta_2}(\delta_\mathbf{2}, \mathbf{u_2}) - {\delta_{\mathbf{2r}}}' \tag{14b}$$

using the fact that the slow variables remain constant in the fast time scale and

$${\delta_{\mathbf{2r}}}' = \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e_z}}{\mathbf{e_z}}' \tag{15}$$

Define a positive-definite decrescent combined Lyapunov function $W_f(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e_z}, \mathbf{e}_{\delta_2}) : [0, \infty) \times D_\mathbf{x} \times D_{\delta_1} \times D_\mathbf{z} \times D_{\delta_2} \to \mathbb{R}$ for the reduced fast subsystem (14) that is continuously differentiable and $D_{\delta_2} \subset \mathbb{R}^{p-l}$ contains the origin

$$W_f(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e_z}, \mathbf{e}_{\delta_2}) = W(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e_z}) + \frac{1}{2}{\mathbf{e}_{\delta_2}}^T \mathbf{e}_{\delta_2} \tag{16}$$

Design $\mathbf{u_2}$ such that the closed-loop reduced fast system (14) satisfies

Condition 6.

$$\frac{\partial W_f}{\partial \mathbf{e_z}}{\mathbf{e_z}}' + \frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}{\mathbf{e}_{\delta_2}}' \leq -\alpha_3 \phi_3^2(\mathbf{e_z}) - \alpha_4 \phi_4^2(\mathbf{e}_{\delta_2}), \quad \alpha_4 > 0$$

B. Stability Analysis

The following theorem summarizes the main result of the paper.

Theorem 1. *Suppose the control* $\mathbf{u}$ *of the system (1) is designed according to the Conditions* $1-14$*. Then for all initial conditions,* $(\mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e}_{\mathbf{z}}, \mathbf{e}_{\delta_2}) \in D_{\mathbf{x}} \times D_{\delta_1} \times D_{\mathbf{z}} \times D_{\delta_2}$*, the control uniformly asymptotically stabilizes the nonlinear singularly perturbed system (1) and equivalently drives the slow state* $\mathbf{x}(t) \to \mathbf{x}_r(t)$ *for all* $\epsilon < \epsilon^*$*, that is defined by the inequality given in (24).*

Proof: The closed-loop complete system in the error coordinates is given as

$$\begin{aligned}
\dot{\mathbf{e}} &= \mathbf{F}(\mathbf{e}, \mathbf{e}_{\mathbf{z}} + \mathbf{z}_r, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) &\text{(17a)}\\
\dot{\mathbf{e}}_{\delta_1} &= \mathbf{f}_{\delta_1}(\mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{u_1}) - \dot{\delta}_{\mathbf{1rC}} &\text{(17b)}\\
\epsilon\dot{\mathbf{e}}_{\mathbf{z}} &= \mathbf{G}(\mathbf{e}, \mathbf{e}_{\mathbf{z}} + \mathbf{z}_{\mathbf{r}}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \epsilon)\\
&- \epsilon\dot{\mathbf{z}}_{\mathbf{r}} &\text{(17c)}\\
\epsilon\dot{\mathbf{e}}_{\delta_2} &= \mathbf{f}_{\delta_2}(\mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \mathbf{u_2}) - \epsilon\dot{\delta}_{\mathbf{2rC}} &\text{(17d)}
\end{aligned}$$

where the subscript 'C' is added to note the difference between (17b)-(17d) and (10) and (15). These expressions for the complete system are noted as

$$\begin{aligned}
\dot{\delta}_{\mathbf{1rC}} &= \frac{\partial \delta_{\mathbf{1r}}}{\partial t} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{e}}\dot{\mathbf{e}} + \\
&\quad \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{x_r}}\dot{\mathbf{x}}_{\mathbf{r}} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \dot{\mathbf{x}}_{\mathbf{r}}}\ddot{\mathbf{x}}_{\mathbf{r}} + \frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{e_z}}\dot{\mathbf{e}}_{\mathbf{z}} &\text{(18)}\\
\dot{\delta}_{\mathbf{2rC}} &= \frac{\partial \delta_{\mathbf{2r}}}{\partial t} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e}}\dot{\mathbf{e}} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{x_r}}\dot{\mathbf{x}}_{\mathbf{r}} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \dot{\mathbf{x}}_{\mathbf{r}}}\ddot{\mathbf{x}}_{\mathbf{r}} + \\
&\quad \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e}_{\delta_1}}\dot{\mathbf{e}}_{\delta_1} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e_z}}\dot{\mathbf{e}}_{\mathbf{z}} &\text{(19)}
\end{aligned}$$

Closed-loop system stability of the system states is analyzed using the composite Lyapunov function approach[10]. Consider a Lyapunov function candidate

$$\nu(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e}_{\mathbf{z}}, \mathbf{e}_{\delta_2}) = V_s(t, \mathbf{e}, \mathbf{e}_{\delta_1}) + W_f(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e}_{\mathbf{z}}, \mathbf{e}_{\delta_2}) \quad (20)$$

for the complete closed-loop system. From the properties of V_s and W_f it follows that $\nu(t, \mathbf{e}, \mathbf{e}_{\delta_1}, \mathbf{e}_{\mathbf{z}}, \mathbf{e}_{\delta_2})$ is positive-definite and decrescent. The derivative of ν along the trajectories of (17) is given by

$$\begin{aligned}
\dot{\nu} &= \frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial \mathbf{e}}\dot{\mathbf{e}} + \frac{\partial V_s}{\partial \mathbf{e}_{\delta_1}}\dot{\mathbf{e}}_{\delta_1} + \frac{\partial W_f}{\partial t} + \frac{\partial W_f}{\partial \mathbf{e}}\dot{\mathbf{e}}\\
&+ \frac{\partial W_f}{\partial \mathbf{e}_{\delta_1}}\dot{\mathbf{e}}_{\delta_1} + \frac{1}{\epsilon}\frac{\partial W_f}{\partial \mathbf{e_z}}\mathbf{e_z}' + \frac{1}{\epsilon}\frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}\mathbf{e}_{\delta_2}' &\text{(21)}
\end{aligned}$$

Note that the vector fields in (17a) and (17c) can also be expressed as

$$\begin{aligned}
&\mathbf{F}(\mathbf{e}, \mathbf{e_z} + \mathbf{z}_r, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) =\\
&\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) + \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}})\\
&- \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) - \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}})\\
&+ \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) \qquad (22)\\
&+ \mathbf{F}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}})\\
&- \mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}})
\end{aligned}$$

$$\begin{aligned}
&\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \epsilon) =\\
&\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}, 0)\\
&+ \mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, 0)\\
&- \mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}, 0)\\
&+ \mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \epsilon)\\
&- \mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, 0) \qquad (23)
\end{aligned}$$

Suppose that Lyapunov functions V_s and W_f also satisfy the following conditions with $\beta_i \geq 0$ and $\gamma_i \geq 0$

Condition 7.

$$\frac{\partial V_s}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) - \frac{\partial V_s}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_{\mathbf{r}}, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \delta_{\mathbf{2r}}) \leq \beta_1\psi_3(\mathbf{e})\phi_4(\mathbf{e}_{\delta_2})$$

Condition 8.

$$\frac{\partial V_s}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) - \frac{\partial V_s}{\partial \mathbf{e}}\mathbf{F}(\mathbf{e}, \mathbf{z_r}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}) \leq \beta_2\psi_3(\mathbf{e})\phi_3(\mathbf{e_z})$$

Condition 9.

$$\frac{\partial V_s}{\partial \mathbf{e}_{\delta_1}}\frac{\partial \delta_{\mathbf{1r}}}{\partial \mathbf{e_z}}\dot{\mathbf{e}}_{\mathbf{z}} \leq \beta_3\psi_3(\mathbf{e})\psi_4(\mathbf{e}_{\delta_1}) + \beta_4\psi_4(\mathbf{e}_{\delta_1})\phi_3(\mathbf{e_z}) + \gamma_1\psi_4^2(\mathbf{e}_{\delta_1})$$

Condition 10.

$$\begin{aligned}
&\frac{\partial W_f}{\partial \mathbf{e_z}}\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x_r}, \dot{\mathbf{x}}_{\mathbf{r}}, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \epsilon) -\\
&\frac{\partial W_f}{\partial \mathbf{e_z}}\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x_r}, \dot{\mathbf{x}}_{\mathbf{r}}, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \mathbf{0})\\
&\leq \epsilon\gamma_2\phi_3^2(\mathbf{e_z}) + \epsilon\beta_5\psi_3(\mathbf{e})\phi_3(\mathbf{e_z}) + \epsilon\beta_6\psi_4(\mathbf{e}_{\delta_1})\phi_3(\mathbf{e_z})\\
&+ \epsilon\beta_7\phi_3(\mathbf{e_z})\phi_4(\mathbf{e}_{\delta_2})
\end{aligned}$$

Condition 11.

$$\begin{aligned}
&\frac{\partial W_f}{\partial t} + \left[\frac{\partial W_f}{\partial \mathbf{e}} - \frac{\partial W_f}{\partial \mathbf{e_z}}\frac{\partial \mathbf{z_r}}{\partial \mathbf{e}}\right]\dot{\mathbf{e}} - \left[\frac{\partial W_f}{\partial \mathbf{e}_{\delta_1}} + \frac{\partial W_f}{\partial \mathbf{e_z}}\frac{\partial \mathbf{z_r}}{\partial \mathbf{e}_{\delta_1}}\right]\dot{\mathbf{e}}_{\delta_1}\\
&- \frac{\partial W_f}{\partial \mathbf{e_z}}\frac{\partial \mathbf{z_r}}{\partial \mathbf{x}_r}\dot{\mathbf{x}}_{\mathbf{r}} - \frac{\partial W_f}{\partial \mathbf{e_z}}\frac{\partial \mathbf{z_r}}{\partial \dot{\mathbf{x}}_r}\ddot{\mathbf{x}}_{\mathbf{r}} \leq \gamma_3\phi_3^2(\mathbf{e_z}) + \beta_8\psi_3(\mathbf{e})\phi_3(\mathbf{e_z})\\
&+ \beta_9\psi_4(\mathbf{e}_{\delta_1})\phi_3(\mathbf{e_z})
\end{aligned}$$

Condition 12.

$$\begin{aligned}
&\frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}\left[\frac{\partial \delta_{\mathbf{2r}}}{\partial t} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e}}\dot{\mathbf{e}} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{x_r}}\dot{\mathbf{x}}_{\mathbf{r}} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \dot{\mathbf{x}}_{\mathbf{r}}}\ddot{\mathbf{x}}_{\mathbf{r}} + \frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e}_{\delta_1}}\dot{\mathbf{e}}_{\delta_1}\right] \leq\\
&\gamma_4\phi_4^2(\mathbf{e}_{\delta_2}) + \beta_{10}\psi_3(\mathbf{e})\phi_4(\mathbf{e}_{\delta_2}) + \beta_{11}\psi_4(\mathbf{e}_{\delta_1})\phi_4(\mathbf{e}_{\delta_2})
\end{aligned}$$

Condition 13.

$$\begin{aligned}
&\frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}\frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e_z}}\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x_r}, \dot{\mathbf{x}}_{\mathbf{r}}, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \epsilon)\\
&- \frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}\frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e_z}}\mathbf{G}(\mathbf{e}, \mathbf{e_z} + \mathbf{z_r}, \mathbf{x_r}, \dot{\mathbf{x}}_{\mathbf{r}}, \mathbf{e}_{\delta_1} + \delta_{\mathbf{1r}}, \mathbf{e}_{\delta_2} + \delta_{\mathbf{2r}}, \mathbf{0})\\
&\leq \epsilon\gamma_5\phi_4^2(\mathbf{e}_{\delta_2}) + \epsilon\beta_{12}\psi_3(\mathbf{e})\phi_4(\mathbf{e}_{\delta_2}) + \epsilon\beta_{13}\psi_4(\mathbf{e}_{\delta_1})\phi_4(\mathbf{e}_{\delta_2})\\
&+ \epsilon\beta_{14}\phi_3(\mathbf{e_z})\phi_4(\mathbf{e}_{\delta_2})
\end{aligned}$$

Condition 14.

$$\frac{\partial W_f}{\partial \mathbf{e}_{\delta_2}}\frac{\partial \delta_{\mathbf{2r}}}{\partial \mathbf{e_z}}[\epsilon\dot{\mathbf{z}}_{\mathbf{r}}] \leq \epsilon\beta_{15}\psi_3(\mathbf{e})\phi_4(\mathbf{e}_{\delta_2}) + \epsilon\beta_{16}\psi_4(\mathbf{e}_{\delta_1})\phi_4(\mathbf{e}_{\delta_2})$$

Conditions $9-14$ enforce restrictions upon the difference between the complete system and the reduced subsystems. Use Conditions $1-14$ into (21) and rearrange to get,

$$\dot{\nu} \leq -\Psi^T \mathbb{K} \Psi \tag{24}$$

where $\Psi = \begin{bmatrix} \psi_3 \\ \psi_4 \\ \phi_3 \\ \phi_4 \end{bmatrix}$ and matrix $\mathbb{K}$ given in (24) is positive-definite for $\epsilon < \epsilon^*$. By definition of the continuous scalar functions ψ_3,ψ_4, ϕ_3 and ϕ_4, it follows that $\dot{\nu}$ is negative definite. By Lyapunov theorem it is concluded that $(\mathbf{e}, \delta_\mathbf{1}, \mathbf{z}, \delta_\mathbf{2}) = (\mathbf{0}, \delta_\mathbf{1r}, \mathbf{z_r}(\mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r), \delta_\mathbf{2r})$ is uniformly asymptotic stable equilibrium of the closed-loop system (17). Further, from definition of the tracking error, it is concluded that $\mathbf{x}(t) \to \mathbf{x}_r(t)$ asymptotically. Since the desired trajectory is assumed to be smooth and bounded with bounded first-order derivatives all the other signals remain bounded for all time. ■

IV. NUMERICAL EXAMPLES

A. Purpose and Scope

The purpose of this section is to illustrate the preceding theoretical developments and demonstrate the controller performance for both standard and non-standard forms of singularly perturbed systems. The first example is taken from Reference [2] and the purpose is to see how the proposed approach compares with composite control technique for standard singularly perturbed systems. The objective of the second example is to analyze the performance and robustness characteristics of the controller for non-standard forms of singularly perturbed systems.

Example 1: Standard Singularly Perturbed Model

The following example is taken from Reference [2]. The objective is to design a regulator to stabilize both the slow and the fast state in the domain $D_x \in [-1, 1]$ and $D_z = [-1/2, 1/2]$.

$$\dot{x} = xz^3; \quad \epsilon\dot{z} = z + u \tag{25}$$

The reduced-order models for the system under study are
Reduced Slow Subsystem

$$\dot{x} = xz^3; \quad 0 = z + u \tag{26}$$

Reduced Fast Subsystem

$$x' = 0; \quad z' = z + u \tag{27}$$

Notice that the algebraic equation in the reduced slow subsystem has an isolated root for the fast state; thus the system given is in standard form.

The controller is designed using the same Lyapunov functions and closed-loop characteristics as in [2]. Using $V(x) = \frac{1}{6}x^6$ as Lyapunov function for the slow subsystem, the desired manifold $z_r = -x^{\frac{4}{3}}$ satisfies Condition 2 with $\alpha_1 = 1$ and $\psi_3(x) = |x|^5$. The control is designed as $u = -3z - 2x^{\frac{4}{3}}$ to satisfy Condition 5 with Lyapunov function $W = \frac{1}{2}(z - z_r)^2$, $\alpha_3 = 2$ and $\phi_3(x, z) = |z - z_r|$. The closed-loop system with $e_z = z - z_r$ becomes

$$\dot{x} = x(e_z + z_r)^3 \tag{28a}$$

$$\epsilon\dot{e}_z = -2e_z + \frac{4}{3}\epsilon x^{\frac{4}{3}}(e_z + z_r)^3 \tag{28b}$$

The inequality in (24) is satisfied for all $\epsilon < 0.4246$.

Notice that the control law designed is exactly same as that obtained using composite control. However, the upper-bound is conservative when compared to the upper-bound obtained using composite control (0.4286). This variation appears because the coefficients of the composite Lyapunov function were chosen as unity in (20) instead of optimal values as in composite control.

Example 2: Non-Standard Singularly Perturbed Model

Consider the following unstable linear system

$$\dot{x} = z - u; \quad \epsilon\dot{z} = x + u \tag{29}$$

The objective is to stabilize the system about $x = 0$ and $z = 0$. Notice that the algebraic equation obtained by setting $\epsilon = 0$ has infinitely many solutions and composite control cannot be applied.

Control Design: With $V(x) = \frac{1}{2}x^2$ as Lyapunov function for the reduced slow subsystem, manifold $z_r = u - \alpha_1 x$ satisfies Condition 2 for $\psi_3(x) = |x|$. Lyapunov function for the fast subsystem is $W(x, z) = \frac{1}{2}(z - z_r)^2$. Condition 5 is satisfied with control $u = -x - \alpha_2(z - z_r)$ and $\phi_3(x, z) = |z - z_r|$. The applied control in original system coordinates is given as

$$u = \frac{-1 - \alpha_1\alpha_2}{1 - \alpha_2}x - \frac{\alpha_2}{1 - \alpha_2}z. \tag{30}$$

Substituting (30) in (29) and with a change of coordinates gives the closed-loop system

$$\dot{x} = -\alpha_1 x + \frac{1}{1 - \alpha_2}e_z \tag{31a}$$

$$\epsilon\dot{e}_z = -\frac{\alpha_2}{1 - \alpha_2}e_z + \epsilon\left[\frac{1 + \alpha_1}{1 - \alpha_2}e_z - \alpha_1(1 + \alpha_1)x\right] \tag{31b}$$

where $e_z = z - z_r$. The constants satisfying Conditions 7-14 are $\beta_2 = \frac{1}{1-\alpha_2}$, $\beta_5 = -\alpha_1(1 + \alpha_1)$ and $\gamma_2 = \frac{1+\alpha_1}{1-\alpha_2}$, rest all being zeros.

Results and Discussion: The closed-loop gains chosen are $\alpha_1 = 0.5$ and $\alpha_2 = 0.5$. With these values, global asymptotic stabilization is satisfied for all $\epsilon < 0.2645$. Table.I documents the closed-loop eigenvalues for different values of ϵ. The closed-loop system loses its time-scale property for $\epsilon > 0.2645$ but remains stable with complex conjugate eigenvalues indicating that the upper bound ϵ^* satisfying condition (24) is conservative. The system becomes unstable for all $\epsilon > 0.4$.

The system given in (29) is the linearized model of the nonlinear non-standard form [11]

$$\dot{x} = \tan z - u; \quad \epsilon\dot{z} = x + u \tag{32}$$

Notice that the fast state appears nonlinearly in the slow dynamics and hence determining a manifold z_r to meet the

$$\mathbb{K} = \begin{bmatrix} \alpha_1 & -\frac{\beta_3}{2} & -\frac{1}{2}\left[\beta_2+\beta_5+\beta_6+\beta_8\right] & -\frac{1}{2}\left[\beta_1-\beta_{10}-\beta_{12}-\beta_{15}\right] \\ -\frac{\beta_3}{2} & \alpha_2-\gamma_1 & -\frac{1}{2}\left[\beta_4+\beta_7+\beta_9\right] & \frac{1}{2}\left[\beta_{11}+\beta_{13}+\beta_{16}\right] \\ -\frac{1}{2}\left[\beta_2+\beta_5+\beta_6+\beta_8\right] & -\frac{1}{2}\left[\beta_4+\beta_7+\beta_9\right] & \frac{\alpha_3}{\epsilon}-\gamma_2-\gamma_3 & \frac{\beta_{14}}{2} \\ -\frac{1}{2}\left[\beta_1-\beta_{10}-\beta_{12}-\beta_{15}\right] & \frac{1}{2}\left[\beta_{11}+\beta_{13}+\beta_{16}\right] & \frac{\beta_{14}}{2} & \frac{\alpha_4}{\epsilon}+\gamma_4+\gamma_5 \end{bmatrix} \quad (24)$$

TABLE I
EXAMPLE 2: CLOSED-LOOP EIGENVALUES

ϵ	Eigenvalues λ
0.05	$\lambda_1 = -0.5914$, $\lambda_2 = -16.9086$
0.1	$\lambda_1 = -0.7396$, $\lambda_2 = -6.7604$
0.2645	$\lambda_{1,2} = -0.6404 \pm 1.167j$
0.35	$\lambda_{1,2} = -0.1786 \pm 1.1818j$
0.4	$\lambda_{1,2} = 0.000 \pm 1.1180j$
0.405	$\lambda_{1,2} = 0.0154 \pm 1.1110j$

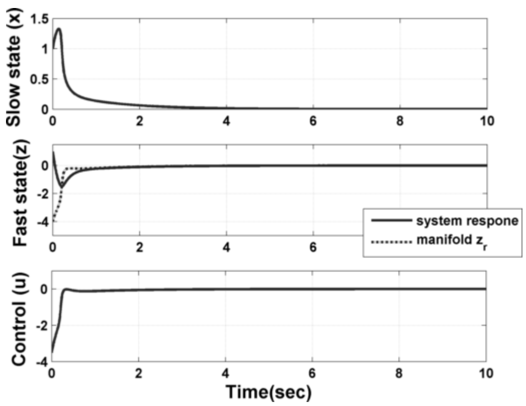

Fig. 1. Example 2: Nonlinear System (32) Closed-loop Response ($\epsilon = 0.1$)

control objective is difficult. Instead, use the controller (30) that was developed for the linear counterpart. The resulting closed-loop system with $\alpha_1 = \alpha_2 = 0.5$ is

$$\dot{x} = 2.5x + \tan z + z; \quad \epsilon\dot{z} = -1.5x - z \quad (33)$$

The controller converts the non-standard form into standard form which uniquely restricts the system onto the desired manifold, which in this case is $z_r = -1.5x$. It is clear that due to the nonlinear nature of the problem the domain of attraction is now restricted to a subspace of the two-dimensional Euclidean space. Using the previously outlined procedure, constants satisfying Conditions $7-14$ are $\beta_2 = 2$ and $\gamma_2 = 3$ with all others being zero in the domain $D_x \in [0, -1)$ and $D_z \in [-1, 2]$. The upper-bound on singular perturbation parameter is computed as $\epsilon^* = 0.2$. Simulation study in this case indicates that stability is maintained for all $\epsilon < 0.4$ and the nonlinear system is asymptotically stabilized in the domain $D_x \in [-2, 2]$ and $D_z \in [-1.5, 2]$. Simulation results for the case of $\epsilon = 0.1$ are shown in Fig.1. Notice that non-zero control is applied until the fast state falls onto the desired manifold.

V. CONCLUSIONS

In this paper, design procedure for tracking the slow states and stabilization of a general class of nonlinear singularly perturbed systems was developed. Based on the stability proof and simulation results presented in the paper, the following conclusions are drawn. First, the control laws computed for standard singularly perturbed systems using composite control[2] are seen to be a special case of the proposed technique. It was also shown that the upper-bound is conservative with comparison to composite control technique as fixed unity gains were used in the formulation of the composite Lyapunov function. These gains can be chosen optimally as done in composite control technique to provide a less conservative upper-bound. Second, simulations for non-standard singularly perturbed systems shows that for all values of $\epsilon < \epsilon^*$ asymptotic convergence is guaranteed and all closed-loop signals remain bounded. The domain of convergence of the proposed technique was seen to be dependent upon the underlying controllers developed for the reduced-order systems. It is possible to guarantee global results by identifying controllers that satisfy Conditions $1-14$ for the complete space spanned by the system states.

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