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# Output feedback control using state observers of a class of nonlinear nonstandard two-time-scale systems

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**ABSTRACT**
This paper develops a theory of output feedback control for a class of nonlinear, nonstandard two-time-scale systems using a controller and a state observer with guaranteed closed-loop stability. Using insights from geometric singular perturbation theory, a sequential controller is designed over two time-scales. For different choices of measurements, Lyapunov-based observer designs are investigated. Both the controller and the observer are designed to guarantee Lyapunov stability of the lower-order reduced subsystems. Using an extension of the composite Lyapunov analysis, it is proved that the full-order nonlinear system with the controller and the observer remains globally asymptotically stable up to a bound of the time-scale separation parameter. In addition, the composite Lyapunov analysis yields sufficient conditions as guidelines to select the gains. The approach and analysis are demonstrated on a nonlinear two-time-scale system for which the reduced subsystems are linear, but the composite Lyapunov analysis handles the nonlinearity present in the full-order dynamics.

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Nonlinear control; singular perturbation; state observers; Lyapunov design

## 1. Introduction

Systems with dynamics evolving in distinct slow and fast time-scales include aircraft (Khalil & Chen, 1990), robotic manipulators (Tavasoli, Eghtesad, & Jafarian, 2009), electrical power systems (Sauer, 2011), chemical reactions (Mélykúti, Hespanha, & Khammash, 2014), production planning in manufacturing (Soner, 1993), and so on. The mathematical model of a two-time-scale system involves a small perturbation parameter $\varepsilon$ such that $0 < \varepsilon << 1$. This parameter $\varepsilon$ signifies how well-separated the two time-scales are. It may be either a function of the system parameters, e.g. in spring-mass-damper, or introduced artificially to distinguish between the slow and the fast states, e.g. in aircraft. Setting this parameter to zero converts the fast dynamics from a set of differential equations to a set of algebraic equations, indicating a singularity in the model. From a physical standpoint, the singularity is due to infinite time-scale separation, i.e. the fast dynamics being infinitely fast. Using this concept, results from the geometric singular perturbation theory (Fenichel, 1979) can be used to design controllers for two-time-scale systems. Controller design using this approach has a few major benefits. First, this results in a nonlinear controller for a nonlinear system; no linear approximation or gain scheduling is involved anywhere in the development. Second, the slow and the fast dynamics are not treated as two completely isolated subsystems; the control design considers the coupling between them in different time-scales. Third, lower-order reduced slow and fast subsystems use a physical insight of the dynamics and at the same time make the control design mathematically easier. According to the geometric singular perturbation theory (Fenichel, 1979), the behaviour of the full-order system can be approximated by the slow subsystem, provided that the fast states can be stabilised on an equilibrium manifold. The fast subsystem describes how the fast states evolve from their initial conditions to their equilibrium trajectory or the manifold. The slow subsystem describes how the slow states evolve from their initial conditions while the fast states stay on the manifold (Khalil, 2002). Even though the controller is designed on reduced subsystems, the stability of the full-order nonlinear system can be proven using composite Lyapunov analysis (Khalil, 2002). Fourth, the exact value of the time-scale separation parameter $\varepsilon$ may not be known for many systems, especially the ones for which $\varepsilon$ is artificially introduced. For static compensation which does not require differentiation with respect to the fast time-scale, the controller design does not require the knowledge of $\varepsilon$, and it still works within a bound of $\varepsilon$ established by the composite Lyapunov analysis.

In the literature, the singular perturbation approach has been used in model order reduction (Grujić, 1979; Liu & Anderson, 1989), control of linear discrete-time systems (Rajagopalan & Naidu, 1980), optimal control problems (Bagagiolo & Bardi, 1998; Forcadel & Rao, 2014; Vigodner, 1997), and so on. In addition, insights from geometric singular perturbation theory have been used for slow state tracking as well as simultaneous slow and fast state tracking of two-time-scale systems (Narang-Siddarth & Valasek, 2014). For slow state regulation or tracking of a two-time-scale system, there is no desired reference for the fast states. Following Tikhonov's theorem (Kokotovic, Khalil, & O'Reilly, 1986), the control should ensure that the fast states can be stabilised on any suitable equilibrium manifold. A unique equilibrium manifold for the fast states can be found for standard singularly perturbed systems. In practice, however, the singularly perturbed models in many important

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applications are nonstandard. For nonstandard systems, there is no unique manifold of the fast states; the manifold is to be either approximated or specified. The method of modified composite control approximates the manifold. It is a two-stage design process in which the control input is considered as a sum of slow and fast controls. Slow and fast controls are chosen using any design method based on the reduced subsystems. The stability of the closed-loop is then determined using Lyapunov analysis. This technique is applied on a generic two-degree-of-freedom nonlinear model as well as a nonlinear six-degree-of-freedom F/A-18A Hornet (Narang-Siddarth & Valasek, 2011a, 2011b). However, the approximation of the manifold often becomes difficult and can be avoided by the method of sequential control. The idea behind sequential control is to use feedback to convert a nonstandard system in the open-loop to a standard system in the closed-loop. It is a Lyapunov design method in which the manifold of the fast state is specified as an intermediate control variable. The manifold is selected such that the reduced slow subsystem is Lyapunov-stable, followed by the selection of the control law such that the reduced fast subsystem is also Lyapunov-stable. Subsequently the composite Lyapunov analysis (Khalil, 2002) gives an upper bound of $\varepsilon$, denoted $\varepsilon^*$, up to which the full-order system is Lyapunov-stable. This method has been extended to four-time-scales such that slow and fast actuator dynamics can be included. A four-time-scale version of the sequential control method was recently applied on a nonlinear six-degree-of-freedom aircraft model (Saha, Valasek, Famularo, & Reza, 2018). This work assumed a deterministic model. A theory to address parametric and state-dependent uncertainties in the fast dynamics was developed in a later work of the authors (Saha, Valasek, & Reza, 2018). While these papers dealt with regulation or tracking of only the slow states using the sequential approach, another challenging problem of simultaneous slow and fast state tracking was addressed using a two-stage design method (Saha & Valasek, 2017).

A major limitation of most control techniques for nonlinear, nonstandard systems with two time-scales is the assumption of full-state feedback. In practice, it is not always possible to measure all of the states, but often some of the states are measurable. Sometimes linear combinations of the states are easier to measure than the individual states themselves. To address this, a theory of output feedback control is needed. In the literature, there are some works on dynamic output feedback control of multiple-time-scale systems using observers. For a robotic manipulator, a linear observer for the fast state was proposed in a few works (Ashayeri, Eghtesad, Farid, & Shabani, 2007; Tavasoli et al., 2009). For a discrete linear two-time-scale system, an output feedback control scheme was proposed based on genetic algorithm (Pan & Chen, 2009; Pan, Pan, & Tsai, 2011). A Linear Quadratic Gaussian (LQG) controller in two time-scales was developed for a missile autopilot (Moghaddam & Zarabadipour, 2012). Observer-based controllers were designed for linear time-delay systems with two-time-scale dynamics (Chiou, 2013). A different work considered observer design for linear two-time-scale systems with Lipschitz constraint (Wang & Liu, 2015). Another work (Hoffmann & Sanders, 1998) presented a two-time-scale observer-based torque control of a induction machine. In all of the above works, the observer design was based on the system dynamics being linear or linearised. For a nonlinear model of an induction motor with two slow and two fast states, Mezouar, Fellah, and Hadjeri (2007, 2008) developed a two-time-scale adaptive sliding mode observer to supplement a regular sliding mode controller in the feedback loop. For a nonlinear two-time-scale spring-mass-damper with one slow and one fast states, both of the controller and the observer were designed in two time-scales (Saha & Valasek, 2016a). This observer-based feedback design was an extension of the method of sequential control (Narang-Siddarth & Valasek, 2014) with the potential to work for nonstandard systems. The Lyapunov design of observers in two time-scales led to guaranteed stability of the corresponding reduced subsystems. A total of six cases – two different cases of dynamics and three difference cases of measurement – were described, and results were presented for four of them. The cases of dynamics were (a) high damping and (b) high stiffness. The cases of measurement were (a) only the slow state measured, (b) only the fast state measured, (c) a linear combination of the states measured. For one of the four cases a subsequent work of the authors (Saha & Valasek, 2016b) extended the procedure to a nonlinear spring-mass-damper with multiple slow and fast states.

While the authors' previous works (Saha & Valasek, 2016a, 2016b) did introduce observer-based output feedback for nonlinear, nonstandard, two-time-scale systems, several aspects of the theory were not fully developed. Stability analysis of the full-order system under output feedback is an important open problem. Ref. (Saha & Valasek, 2016a) shows Lyapunov stability of the individual reduced subsystems. These lower-order subsystems are constructed on the assumption of infinite time-scale separation. However, time-scale separation for the original full-order system is finite. Therefore, it must be investigated if the controller and the observer designed on the reduced subsystems will preserve the stability of the full-order system. For full-state feedback, it is well-established that the controller can accommodate finite time-scale separation, i.e. the parameter $\varepsilon$ taking on values up to a certain $\varepsilon^*$. This $\varepsilon^*$ is a function of the system parameters and gains, and it can be determined from the composite Lyapunov analysis. However, for output feedback an equivalent upper bound $\varepsilon^{**}$ was not available in the closed form. Ref. (Saha & Valasek, 2016b) shows a way to extend the composite Lyapunov analysis for output feedback, but the bound of time-scale separation is determined numerically by plotting all the eigenvalues of a matrix. This numerical approach misses an important insight: it is not possible to understand which gain or system parameter changes the stability bound, and by how much. This requires a more rigorous analytical treatment of the composite Lyapunov analysis for output feedback. Moreover, the authors' previous work (Saha & Valasek, 2016a) considers a specific type of nonlinearity, and the exact expression of the nonlinear function is used by both the controller and the observer. It is to be investigated if this method can address a more generic family of nonlinear functions with only a few known properties. In addition, in the previous work, it was not possible to develop Lyapunov-based observers for all the cases of measurement. In particular, the case of only the fast state being measured was shown to be mathematically intractable for observer design. It is possible that for a more generic system there may be a way to design an observer when the measurement is only the fast state.

The current work addresses the issues discussed above by developing a theory of output feedback control for a more generic class of nonlinear, nonstandard, two-time-scale systems. The class of systems considered in this paper can represent mechanical systems with displacement being the slow state and velocity being the fast state. The fast dynamics are comprised of three parts: one part linear in the states, one part nonlinear in the states, and one part being the control input. The overall nonlinearity is a sum of two terms, and each term is a product of two nonlinear functions: a sector-bounded nonlinear function in one state, and a bounded in magnitude nonlinear function in another state. The sector-bounded function can capture nonlinear behaviours which may become large as the state becomes large. On the other hand, the nonlinear function bounded in magnitude can describe hard nonlinearities such as saturation. The fast dynamics are such that the nonlinearity is present in the full-order system but not present in the reduced subsystems.

This paper makes a major contribution for this new class of nonlinear systems. Output feedback controllers using state observers are developed with guarantees of stability for all of the three cases: (i) only the slow state is measured, (ii) only the fast state is measured, and (iii) a linear combination of the states is measured. For each case the controller and the observer structures are selected using Lyapunov design on the lower-order reduced subsystems. The gains in the control law and observer dynamics are dictated by the bounds of the nonlinear functions. The observer operates in the fast time-scale, and estimates one or both of the states depending on the measurement. An extension of the existing composite Lyapunov analysis is performed for all three cases. For each case this analysis produces a closed-form upper bound $\varepsilon^{**}$ of time-scale separation $\varepsilon$ guaranteeing global asymptotic stability. In addition, this analysis also yields constraints on the gains so they can be selected for guaranteed stability.

This paper is organised as follows. The two-time-scale nonlinear model is described in Section 2. The control law development and the stability analysis of the full-order system are presented in Section 3. Time-histories and numerical stability bounds are in Section 4. Major conclusions of the current work are in Section 5.

## 2. The two-time-scale model

This paper develops a theory of output feedback control for the following class of nonlinear, nonstandard, two-time-scale systems:

$$\begin{aligned}\dot{x} &= z\\ \varepsilon\dot{z} &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + u.\end{aligned} \tag{1}$$

Equation (1) refers to a class of second-order systems with the displacement $x$ being the slow state and the velocity $z$ being the fast state. The time-scale separation parameter $\varepsilon$ satisfying $0 < \varepsilon << 1$ represents how fast $z$ is evolving compared to $x$. The time-evolution of the fast state $z$ is dictated by three parts: one part $(px + qz)$ is linear in the states; a second part $\varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z))$ is nonlinear in the states, and a third part is the control input $u$.

The parameters $p$ and $q$ are known. The function $f_1(x)$ can represent *any* sector-bounded nonlinearity in the slow state $x$, contained between the line segments $-F_1|x|$ and $F_1|x|$. Similarly, $g_2(z)$ can be *any* sector-bounded nonlinearity in the fast state $z$, contained between the line segments $-G_2|z|$ and $G_2|z|$. On the other hand, either of $g_1(z)$ and $f_2(x)$ can be *any* bounded nonlinearity contained in $[-G_1, G_1]$ and $[-F_2, F_2]$, respectively. For the purpose of control law development, no specific form is assumed for any of the nonlinear functions, but the sectors or magnitude-bounds $F_1, G_1, F_2, G_2$ are assumed to be known constants. The nonlinearity $f_1(x)g_1(z) + f_2(x)g_2(z)$ being multiplied by $\varepsilon$ means that this is present in the full-order dynamics, but will not show up in the reduced-order dynamics obtained by the substitution $\varepsilon = 0$.

## 3. Control law development

It is desired for the control to drive the slow state $x$ from its initial condition $x(0)$ to the origin. The fast state $z$ has an initial condition $z(0)$, but does not have any specified reference. According to the singular perturbation theory, the control $u$ must be able to stabilise $z$ about any suitable equilibrium manifold $z^0$ in the fast time-scale, such that the slow state $x$ can be regulated in the slow time-scale. The method of sequential control (Narang-Siddarth & Valasek, 2014) was developed to accomplish this objective. However, this method assumes full-state feedback. In case all the states are not explicitly measured, one way to design output feedback control is to use a state observer to feed estimates of the unmeasured states to the controller. Since the system under study is nonlinear, and the controller and observer designs are likely to be coupled, a new theory needs to be developed in order to establish closed-loop stability under output feedback. To be able to compare the existing full-state feedback design and the new output feedback design, this Section first addresses the design of the full-state feedback controller. Subsequently, the output feedback design is discussed, and it includes three cases of measurement: (a) only the slow state is measured, (b) only the fast state is measured, (c) a linear combination of the states is measured.

### 3.1 Full-state feedback controller design

The method of sequential control involves design of the manifold and control to ensure the Lyapunov stability of the reduced subsystems, and composite Lyapunov analysis to establish the Lyapunov stability of the full-order system. In this paper, composite Lyapunov analysis for full-state feedback is performed so the result can be compared with that for output feedback.

#### 3.1.1 Design of manifold and control

The first step in controller design using reduced subsystems is the design of the manifold such that the reduced slow subsystem is stabilised about $x = 0$. Substituting $\varepsilon = 0$ in the full-order dynamics (1) the reduced slow subsystem is

$$\begin{aligned}\dot{x} &= z^0\\ 0 &= px + qz^0 + u^0\end{aligned} \tag{2}$$

where $z^0$ is the manifold of the fast state to be designed, and $u^0$ is the effective control in the slow time-scale. For this subsystem, a positive-definite candidate Lyapunov function and its derivative with respect to the slow time-scale $t$ are

$$\begin{aligned} V_{s_c} &= \tfrac{1}{2}x^2 \\ \dot{V}_{s_c} &= x\dot{x}. \end{aligned} \tag{3}$$

Let $f(.)|_{(i)}$ denote the value of the function $f(.)$ for system denoted by equation ($i$). For the reduced slow subsystem (2), the time-derivative of the Lyapunov function $V_{s_c}$ becomes

$$\dot{V}_{s_c}|_{(2)} = x\dot{x}|_{(2)} = xz^0. \tag{4}$$

Choose the manifold as

$$z^0 = -k_1 x \tag{5}$$

where $k_1 > 0$ is a gain, such that the time-derivative of the Lyapunov function

$$\dot{V}_{s_c}|_{(2)} = -k_1 x^2 \tag{6}$$

is negative-definite. Thus the equilibrium $x = 0$ is Lyapunov-stable.

The second step is to design the control $u$ such that the reduced fast subsystem is stabilised about the manifold $z^0$ selected in the first step. Construct the fast time-scale $\tau = t/\varepsilon$. In this time-scale, the full-order system (1) becomes

$$\begin{aligned} x' &= \varepsilon z \\ z' &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + u. \end{aligned} \tag{7}$$

The 'prime' denotes differentiation with respect to the fast time-scale $\tau$. Set $\varepsilon = 0$ in (7) to obtain the reduced fast subsystem

$$\begin{aligned} x' &= 0 \\ z' &= px + qz + u. \end{aligned} \tag{8}$$

For this subsystem, a positive-definite candidate Lyapunov function and its derivative with respect to the fast time-scale $\tau$ are

$$\begin{aligned} V_{f_c} &= \tfrac{1}{2}(z - z^0)^2 \\ V'_{f_c} &= (z - z^0)(z' - z^{0\prime}). \end{aligned} \tag{9}$$

For the reduced fast subsystem (8), the time-derivative becomes

$$\begin{aligned} V'_{f_c}|_{(8)} &= (z - z^0)(z'|_{(8)} - z^{0\prime}|_{(8)}) \\ &= (z - z^0)(z'|_{(8)} + k_1 x'|_{(8)}) = (z - z^0)(px + qz + u). \end{aligned} \tag{10}$$

Design the control as

$$u = -(px + qz) - k_2(z - z^0) = -(p + k_1 k_2)x - (q + k_2)z \tag{11}$$

where $k_2 > 0$ is another gain, such that the time-derivative

$$V'_{f_c}|_{(8)} = -k_2(z - z^0)^2 \tag{12}$$

is negative-definite. Thus the equilibrium $z = z^0$ for the reduced fast subsystem is Lyapunov-stable.

It is important to note that although the control is designed using the reduced subsystems given by (2) and (8), the control is to be implemented on the full-order system represented by (1) or equivalently by (7). It is also important to note that the full-order system is nonlinear, while the reduced subsystems are linear. The difference between the full-order and reduced-order dynamics and the proof that the control law ensures stability of the full-order system up to a certain bound of time-scale separation are addressed in the next section.

### *3.1.2 Stability of the full-order system: composite Lyapunov analysis*

The composite Lyapunov analysis (Khalil, 2002) starts with selecting a candidate Lyapunov function for the full-order nonlinear system, and yields an upper bound $\varepsilon^*$ of the perturbation parameter $\varepsilon$, up to which the time-derivative of this Lyapunov function is negative-definite. Using a composite of the two individual Lyapunov functions used for controller design, a candidate Lyapunov function for the full-order system is

$$V_{c_{fs}} = w_1 V_{s_c} + w_2 V_{f_c}. \tag{13}$$

This function is positive-definite and radially unbounded for any $w_1, w_2 > 0$. The factors $w_1, w_2 > 0$ are gains signifying the contributions of the individual Lyapunov functions to the composite. The following theorem gives the bound of the time-scale separation parameter $\varepsilon$ for stability of the full-order system under full-state feedback.

**Theorem 3.1:** *For any $k_1 > 0, k_2 > 0, w_1 > 0, w_2 > 0$, the full-state feedback control law* (11) *keeps the equilibrium $x = 0, z = z^0$ of the full-order nonlinear system* (1) *(or equivalently* (7)*) globally asymptotically stable, and therefore $z \to z^0$, $x \to 0$ as $t \to \infty$ from any set of initial conditions $z(0), x(0)$ for $0 < \varepsilon < \varepsilon^*$, where*

$$\varepsilon^* = \frac{4w_1 w_2 k_1 k_2}{[|w_2 k_1^2 - w_1| + w_2(F_1 G_1 + F_2 G_2 k_1)]^2 + 4w_1 w_2 k_1(k_1 + F_2 G_2)}. \tag{14}$$

***Proof:*** The time-derivative of the composite Lyapunov function (13) for the full-order system (1) (equivalently (7)) is

$$\dot{V}_{c_{fs}} = w_1 \dot{V}_{s_c}|_{(1)} + \frac{w_2}{\varepsilon} V'_{f_c}|_{(7)}. \tag{15}$$

Adding and subtracting the time-derivatives of Lyapunov functions for appropriate reduced subsystems, the time-derivative of the composite Lyapunov function for the full-order system becomes

$$\begin{aligned} \dot{V}_{c_{fs}} &= w_1 \dot{V}_{s_c}|_{(2)} + \frac{w_2}{\varepsilon} V'_{f_c}|_{(8)} + w_1(\dot{V}_{s_c}|_{(1)} - \dot{V}_{s_c}|_{(2)}) \\ &\quad + \frac{w_2}{\varepsilon}(V'_{f_c}|_{(7)} - V'_{f_c}|_{(8)}). \end{aligned} \tag{16}$$

The first two terms in the right-hand side of Equation (16) correspond to the reduced-order dynamics. The third and the fourth terms correspond to the difference between the full-order and the reduced-order dynamics. Making appropriate substitutions for the time-derivatives of the Lyapunov functions, Equation (16) becomes

$$\dot{V}_{c_{fs}} = -w_1k_1x^2 - \frac{w_2}{\varepsilon}k_2(z-z^0)^2 + w_1x(\dot{x}|_{(1)} - \dot{x}|_{(2)}) + \frac{w_2}{\varepsilon}(z-z^0)[(z'|_{(7)} - z'|_{(8)}) + k_1(x'|_{(7)} - x'|_{(8)})]. \tag{17}$$

Substituting for all the relevant dynamics terms, the time-derivative of the composite Lyapunov function reduces to

$$\dot{V}_{c_{fs}} = -w_1k_1x^2 - \frac{w_2}{\varepsilon}k_2(z-z^0)^2 + w_1x(z-z^0) + \frac{w_2}{\varepsilon}(z-z^0)[\varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z) + k_1\varepsilon z]. \tag{18}$$

Notice that $z(z-z^0) = (z-z^0+z^0)(z-z^0) = (z-z^0-k_1x)(z-z^0) = (z-z^0)^2 - k_1x(z-z^0)$. Therefore,

$$\dot{V}_{c_{fs}} = -w_1k_1x^2 - \frac{w_2}{\varepsilon}k_2(z-z^0)^2 + w_1x(z-z^0) + w_2k_1(z-z^0)^2 - w_2k_1^2x(z-z^0) + w_2(z-z^0)f_1(x)g_1(z) + w_2(z-z^0)f_2(x)g_2(z). \tag{19}$$

Now,

$$(z-z^0)f_1(x)g_1(z) \le |z-z^0||f_1(x)||g_1(z)| \le |z-z^0|F_1|x|G_1 = F_1G_1|x||z-z^0| \tag{20}$$

and

$$\begin{aligned}(z-z^0)f_2(x)g_2(z) &\le |z-z^0||f_2(x)||g_2(z)| \le |z-z^0|F_2G_2|z| \\ &= F_2G_2|z-z^0||z-z^0+z^0| \\ &= F_2G_2|z-z^0||z-z^0-k_1x| \\ &\le F_2G_2|z-z^0|[|z-z^0| + k_1|x|] \\ &= F_2G_2(z-z^0)^2 + F_2G_2k_1|x||z-z^0|.\end{aligned} \tag{21}$$

Using (20) and (21), the time-derivative (19) can be written as

$$\dot{V}_{c_{fs}} \le -w_1k_1x^2 - w_2\left(\frac{k_2}{\varepsilon} - k_1\right)(z-z^0)^2 - (w_2k_1^2 - w_1)x(z-z^0) + w_2(F_1G_1|x||z-z^0| + F_2G_2(z-z^0)^2 + F_2G_2k_1|x||z-z^0|). \tag{22}$$

Notice that $-(w_2k_1^2 - w_1)x(z-z^0) \le |w_2k_1^2 - w_1||x||z-z^0|$. Hence,

$$\dot{V}_{c_{fs}} \le -w_1k_1x^2 - w_2\left(\frac{k_2}{\varepsilon} - k_1 - F_2G_2\right)(z-z^0)^2 + (|w_2k_1^2 - w_1| + w_2(F_1G_1 + F_2G_2k_1))|x||z-z^0|. \tag{23}$$

Inequality (23) can be written in the following vector-matrix form:

$$\dot{V}_{c_{fs}} \le -\mathbb{X}^{\mathrm{T}}\mathbb{K}\mathbb{X} \tag{24}$$

where $\mathbb{X} := \begin{bmatrix} |x| \\ |z-z^0| \end{bmatrix}$ and

$$\mathbb{K} := \begin{bmatrix} w_1k_1 & -\frac{1}{2}(|w_2k_1^2 - w_1| + w_2(F_1G_1 + F_2G_2k_1)) \\ -\frac{1}{2}(|w_2k_1^2 - w_1| + w_2(F_1G_1 + F_2G_2k_1)) & w_2\left(\frac{k_2}{\varepsilon} - k_1 - F_2G_2\right) \end{bmatrix}.$$

If $\mathbb{K}$ is positive-definite, the time-derivative of the composite Lyapunov function is negative-definite everywhere in the state-space, and thus the equilibrium $x = 0, z = z^0$ is guaranteed to be globally asymptotically stable. $\mathbb{K}_{2\times2}$ is positive-definite if its $1\times1$ and $2\times2$ Leading Principal Minors (LPMs) are positive. The $1\times1$ LPM of $\mathbb{K}$ is $w_1k_1$ which is positive since $w_1 > 0$ and $k_1 > 0$ by design. The $2\times2$ LPM is positive if

$$w_1k_1w_2\left(\frac{k_2}{\varepsilon} - k_1 - F_2G_2\right) > \frac{1}{4}(|w_2k_1^2 - w_1| + w_2(F_1G_1 + F_2G_2k_1))^2 \tag{25}$$

Solving for $\varepsilon$, one obtains $0 < \varepsilon < \varepsilon^*$, where $\varepsilon^*$ is the upper bound given by (14). This completes the proof. ■

### 3.2 Output feedback using state observers

For the two-time-scale system (1), the unavailability of full-state feedback means that the measurement can be either the slow state $x$, or the fast state $z$, or a combination of $x$ and $z$. This paper investigates observer designs for all three cases: (a) only the slow state is measured, i.e. the output equation is $y = x$, (b) only the fast state is measured, i.e. the output equation is $y = z$, (c) a linear combination of the states is measured, i.e. the output equation is $y = c_1x + c_2z$, where $c_1$ and $c_2$ are nonzero constants. The following three sections develop the theory of observer design and stability analysis under output feedback for these three cases.

#### 3.2.1 Slow state measured

For this case, the system parameter $q$ is assumed to be negative. The full-order system with the state and output equations in the slow time-scale is

$$\begin{aligned}\dot{x} &= z \\ \varepsilon\dot{z} &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + \bar{u} \\ y &= x.\end{aligned} \tag{26}$$

In the fast time-scale $\tau = \frac{t}{\varepsilon}$, the full-order system is

$$\begin{aligned}x' &= \varepsilon z \\ z' &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + \bar{u} \\ y &= x.\end{aligned} \tag{27}$$

The output feedback control is denoted $\bar{u}$ to distinguish it from the full-state feedback control $u$. Since the slow state $x$ is measured but the fast state $z$ is not, the control law is modified from the full-state version $u$ given by (11) to

$$\bar{u} = -(p + k_1 k_2)x - (q + k_2)\hat{z} \quad (28)$$

with the fast state $z$ replaced by its estimate $\hat{z}$. The state $z$ evolves in the fast time-scale. Consequently, an observer needs to be designed to make the estimate $\hat{z}$ converge to $z$ in the fast time-scale. The observer is designed based on the reduced fast subsystem

$$\begin{aligned} x' &= 0 \\ z' &= px + qz + \bar{u} \\ y &= x. \end{aligned} \quad (29)$$

Assume the observer dynamics to be a function of the observed fast state, control and output, of the form

$$\hat{z}' = \phi(\hat{z}, \bar{u}, y). \quad (30)$$

A positive-definite observer Lyapunov function candidate and its derivative with respect to the fast time-scale $\tau$ are

$$\begin{aligned} V_{f_o} &= \tfrac{1}{2}(z - \hat{z})^2 \\ V'_{f_o} &= (z - \hat{z})(z' - \hat{z}'). \end{aligned} \quad (31)$$

For the reduced fast subsystem (29), the time-derivative becomes

$$\begin{aligned} V'_{f_o}|_{(29)} &= (z - \hat{z})(z'|_{(29)} - \hat{z}'|_{(30)}) \\ &= (z - \hat{z})(px + qz + \bar{u} - \phi(.)). \end{aligned} \quad (32)$$

Choose observer dynamics

$$\phi(.) = py + q\hat{z} + \bar{u} \quad (33)$$

such that the time-derivative of the observer Lyapunov function becomes

$$V'_{f_o}|_{(29)} = q(z - \hat{z})^2 = -\bar{q}(z - \hat{z})^2 \quad (34)$$

where $\bar{q} := -q$. This time-derivative is negative-definite since $q$ was assumed to be negative. Thus the equilibrium $\hat{z} = z$ is asymptotically stable, and thus the observed fast state $\hat{z}$ converges to the actual fast state $z$ evolving according to the reduced fast dynamics (29).

An extension of the composite Lyapunov analysis is performed to account for the difference between the full-order and the reduced-order dynamics with the observer included. Similar to the analysis for full-state feedback, the objective here is to find a *new* upper bound $\varepsilon^{**}$ of the perturbation parameter $\varepsilon$ such that Lyapunov stability of full-order nonlinear system is guaranteed in the range $0 < \varepsilon < \varepsilon^{**}$. A candidate composite Lyapunov function for the full-order system is

$$V_{c_{ob}} = \alpha_1 V_{s_c} + \alpha_2 V_{f_c} + \alpha_3 V_{f_o}. \quad (35)$$

This function is a weighted sum of the two Lyapunov functions used for the controller design and one Lyapunov function used for the observer design. The gains $\alpha_1, \alpha_2, \alpha_3$ represent the weights of the individual Lyapunov functions in the composite. The composite Lyapunov function is positive-definite and radially unbounded for any $\alpha_1, \alpha_2, \alpha_3 > 0$. The following theorem gives the bound of $\varepsilon$ for which the full-order system (26) is guaranteed to be globally asymptotically stable.

**Theorem 3.2:** *Suppose that the system parameter $q < 0$. Let $\bar{q} := -q$. Suppose that the gains $k_1 > 0, k_2 > 0, \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ can be selected such that the following inequalities hold:*

$$\begin{aligned} &2\alpha_1 k_1 > |\alpha_2 k_1^2 - \alpha_1| + (\alpha_2 + \alpha_3)(F_1 G_1 + F_2 G_2 k_1) \\ &k_2 > \frac{\bar{q}}{3} \\ &2\alpha_3 \bar{q} > \alpha_2 |k_2 - \bar{q}| \end{aligned} \quad (36)$$

*then the output feedback control law (28) supported by the observer (33) keeps the equilibrium $x = 0, z = z^0, \hat{z} = z$ of the full-order nonlinear system (26), (33) globally asymptotically stable, and therefore $\hat{z} \to z$, $z \to z^0$, $x \to 0$ as $t \to \infty$ from any set of initial conditions $\hat{z}(0), z(0), x(0)$ for $0 < \varepsilon < \varepsilon^{**}$ where $\varepsilon^{**} = \min(\varepsilon_1^{**}, \varepsilon_2^{**})$ with*

$$\begin{aligned} \varepsilon_1^{**} &= \frac{\alpha_2(2k_2 - |k_2 - \bar{q}|)}{2\alpha_2(k_1 + F_2 G_2) + |\alpha_2 k_1^2 - \alpha_1| + \alpha_2(F_1 G_1 + F_2 G_2 k_1) + \alpha_3 F_2 G_2} \\ \varepsilon_2^{**} &= \frac{2\alpha_3 \bar{q} - \alpha_2 |k_2 - \bar{q}|}{\alpha_3(F_1 G_1 + F_2 G_2(1 + k_1))} \end{aligned} \quad (37)$$

***Proof:*** The time-derivative of the composite Lyapunov function (35) for the full-order system is

$$\dot{V}_{c_{ob}} = \alpha_1 \dot{V}_{s_c}|_{(26)} + \frac{\alpha_2}{\varepsilon} V'_{f_c}|_{(27)} + \frac{\alpha_3}{\varepsilon} V'_{f_o}|_{(27)}. \quad (38)$$

Adding and subtracting the time-derivatives of Lyapunov functions for the appropriate reduced subsystems, the time-derivative of the composite Lyapunov function becomes

$$\begin{aligned} \dot{V}_{c_{ob}} &= \alpha_1 \dot{V}_{s_c}|_{(2)} + \frac{\alpha_2}{\varepsilon} V'_{f_c}|_{(8)} + \frac{\alpha_3}{\varepsilon} V'_{f_o}|_{(29)} \\ &\quad + \alpha_1(\dot{V}_{s_c}|_{(26)} - \dot{V}_{s_c}|_{(2)}) \\ &\quad + \frac{\alpha_2}{\varepsilon}(V'_{f_c}|_{(27)} - V'_{f_c}|_{(8)}) + \frac{\alpha_3}{\varepsilon}(V'_{f_o}|_{(27)} - V'_{f_o}|_{(29)}). \end{aligned} \quad (39)$$

The first two terms on the right-hand side of Equation (39) correspond to the two reduced subsystems used for controller design. The third term corresponds to reduced subsystem used for observer design. The next three terms correspond to the difference between the full-order and the reduced-order dynamics. Substituting for appropriate Lyapunov functions and their derivatives for the corresponding reduced subsystems leads to

$$\begin{aligned} \dot{V}_{c_{ob}} &= -\alpha_1 k_1 x^2 - \frac{\alpha_2}{\varepsilon} k_2 (z - z^0)^2 - \frac{\alpha_3}{\varepsilon} \bar{q}(z - \hat{z})^2 \\ &\quad + \alpha_1 x(\dot{x}|_{(26)} - \dot{x}|_{(2)}) \end{aligned}$$

$$+\frac{\alpha_2}{\varepsilon}(z-z^0)[(z'|_{(27)}-z'|_{(8)})-(z^{0'}|_{(27)}-z^{0'}|_{(8)})]$$
$$+\frac{\alpha_3}{\varepsilon}(z-\hat{z})[(z'|_{(27)}-z'|_{(29)})-(\hat{z}'|_{(27)}-\hat{z}'|_{(29)})]. \tag{40}$$

Substituting for all the dynamics terms,

$$\begin{aligned}\dot{V}_{c_{ob}} &= -\alpha_1 k_1 x^2 - \frac{\alpha_2}{\varepsilon}k_2(z-z^0)^2 \\ &\quad - \frac{\alpha_3}{\varepsilon}\bar{q}(z-\hat{z})^2 + \alpha_1 x(z-z^0) \\ &\quad + \frac{\alpha_2}{\varepsilon}(z-z^0)[\varepsilon(f_1(x)g_1(z) \\ &\quad + f_2(x)g_2(z)) + \bar{u} - u + k_1\varepsilon z] \\ &\quad + \frac{\alpha_3}{\varepsilon}(z-\hat{z})[\varepsilon(f_1(x)g_1(z)+f_2(x)g_2(z))]. \end{aligned} \tag{41}$$

Using Equations (11) and (28), the difference between the output feedback control $\bar{u}$ and the full-state feedback control $u$ can be expressed as

$$\bar{u}-u=(k_2-\bar{q})(z-\hat{z}). \tag{42}$$

Upper bounds of the terms $(z-z^0)f_1(x)g_1(z)$ and $(z-z^0)f_2(x)g_2(z)$ were found in the proof of Theorem 3.1. They are given by (20) and (21), respectively. The other two terms $(z-\hat{z})f_1(x)g_1(z)$ and $(z-\hat{z})f_2(x)g_2(z)$ can be bounded as follows.

$$\begin{aligned}(z-\hat{z})f_1(x)g_1(z) &\le |z-\hat{z}||f_1(x)||g_1(z)| \\ &\le |z-\hat{z}|F_1|x|G_1 = F_1G_1|x||z-\hat{z}| \end{aligned} \tag{43}$$

and

$$\begin{aligned}(z-\hat{z})f_2(x)g_2(z) &\le |z-\hat{z}||f_2(x)||g_2(z)| \le |z-\hat{z}|F_2G_2|z| \\ &= F_2G_2|z-\hat{z}||z-z^0+z^0| \\ &= F_2G_2|z-\hat{z}||z-z^0-k_1x| \\ &\le F_2G_2|z-\hat{z}|(|z-z^0|+k_1|x|) \\ &= F_2G_2|z-z^0||z-\hat{z}|+F_2G_2k_1|x||z-\hat{z}| \end{aligned} \tag{44}$$

Using (42), (20), (21), (43), (44) and expressing $z(z-z^0)=(z-z^0)^2-k_1x(z-z^0)$, Equation (41) becomes

$$\begin{aligned}\dot{V}_{c_{ob}} &\le -\alpha_1k_1x^2 - \frac{\alpha_2}{\varepsilon}k_2(z-z^0)^2 - \frac{\alpha_3}{\varepsilon}\bar{q}(z-\hat{z})^2 \\ &\quad + \alpha_1x(z-z^0) + \alpha_2k_1(z-z^0)^2 \\ &\quad - \alpha_2k_1^2x(z-z^0) + \frac{\alpha_2}{\varepsilon}(k_2-\bar{q})(z-z^0)(z-\hat{z}) \\ &\quad + \alpha_2F_1G_1|x||z-z^0| \\ &\quad + \alpha_2F_2G_2(z-z^0)^2 + \alpha_2F_2G_2k_1|x||z-z^0| \\ &\quad + \alpha_3F_1G_1|x||z-\hat{z}| \\ &\quad + \alpha_3F_2G_2|z-z^0||z-\hat{z}| + \alpha_3F_2G_2k_1|x||z-\hat{z}| \end{aligned} \tag{45}$$

The terms involving $x(z-z^0)$ and $(z-z^0)(z-\hat{z})$ can be bounded as

$$\begin{aligned}(\alpha_2k_1^2-\alpha_1)x(z-z^0) &\le |\alpha_2k_1^2-\alpha_1||x||z-z^0| \\ \frac{\alpha_2}{\varepsilon}(k_2-\bar{q})(z-z^0)(z-\hat{z}) &\le \frac{\alpha_2}{\varepsilon}|k_2-\bar{q}||z-z^0||z-\hat{z}| \end{aligned} \tag{46}$$

Using (46) and collecting coefficients,

$$\begin{aligned}\dot{V}_{c_{ob}} &\le -\alpha_1k_1x^2 - \alpha_2\left(\frac{k_2}{\varepsilon}-k_1-F_2G_2\right)(z-z^0)^2 \\ &\quad - \frac{\alpha_3}{\varepsilon}\bar{q}(z-z^0)^2 \\ &\quad + (|\alpha_2k_1^2-\alpha_1| + \alpha_2(F_1G_1+F_2G_2k_1))|x||z-z^0| \\ &\quad + \left(\alpha_3F_2G_2+\frac{\alpha_2}{\varepsilon}|k_2-\bar{q}|\right)|z-z^0||z-\hat{z}| \\ &\quad + \alpha_3(F_1G_1+F_2G_2k_1)|x||z-\hat{z}| \end{aligned} \tag{47}$$

Applying completion of squares, i.e. $|a||b| \le \frac{1}{2}(a^2+b^2)$ to the product terms $|x|\,|z-z^0|$, $|z-z^0|\,|z-\hat{z}|$, $|x|\,|z-\hat{z}|$, the time-derivative (47) can be expressed in the following vector-matrix form:

$$\dot{V}_{c_{ob}} \le -\bar{\mathbb{X}}^{\mathrm{T}}\bar{\mathbb{K}}\bar{\mathbb{X}} \tag{48}$$

where $\bar{\mathbb{X}} := \begin{bmatrix} x \\ z-z^0 \\ z-\hat{z} \end{bmatrix}$ and $\bar{\mathbb{K}} := \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}$, with the diagonal elements $k_{11}, k_{22}, k_{33}$ given by

$$\begin{aligned} k_{11} &= \alpha_1k_1 - \frac{1}{2}\left(|\alpha_2k_1^2-\alpha_1| + (\alpha_2+\alpha_3)(F_1G_1+F_2G_2k_1)\right) \\ k_{22} &= \alpha_2\left(\frac{k_2}{\varepsilon}-k_1-F_2G_2\right) - \frac{1}{2}\Bigg[|\alpha_2k_1^2-\alpha_1| \\ &\quad + \alpha_2\left(\frac{k_2-\bar{q}}{\varepsilon}+F_1G_1+F_2G_2k_1\right)+\alpha_3F_2G_2\Bigg] \\ k_{33} &= \frac{\alpha_3}{\varepsilon}\bar{q} - \frac{\alpha_2|k_2-\bar{q}|}{2\varepsilon} - \frac{1}{2}\alpha_3(F_1G_1+F_2G_2(1+k_1)) \end{aligned} \tag{49}$$

If all the diagonal elements of the matrix $\bar{\mathbb{K}}$ are positive, the time-derivative of the composite Lyapunov function will be negative-definite everywhere in the state-space, and hence the equilibrium $x=0, z=z^0, \hat{z}=z$ of the full-order nonlinear system will be globally asymptotically stable. The element $k_{11}$ is positive if the first sufficient condition in (36) holds. The element $k_{22}$ is positive if $0<\varepsilon<\varepsilon_1^{**}$, where $\varepsilon_1^{**}$ is given by (37). The upper bound $\varepsilon_1^{**}$ is positive if the second sufficient condition in (36) holds. The element $k_{33}$ is positive if $0<\varepsilon<\varepsilon_2^{**}$ where $\varepsilon_2^{**}$ is given by (37). The upper bound $\varepsilon_2^{**}$ is positive if the third sufficient condition in (36) holds. Therefore, if all the three sufficient conditions in (36) hold, global asymptotic stability holds for $0<\varepsilon<\varepsilon^{**}$ where $\varepsilon^{**}=\min(\varepsilon_1^{**},\varepsilon_2^{**})$. This completes the proof. ■

### *3.2.2 Fast state measured*

For this case, the system parameter $p$ is assumed to be nonzero. The full-order system with the state and output equations in the slow time-scale is

$$\begin{aligned}\dot{x} &= z\\ \varepsilon\dot{z} &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + \bar{u}\\ y &= z\end{aligned} \tag{50}$$

In the fast time-scale, the full-order system is

$$\begin{aligned}x' &= \varepsilon z\\ z' &= \varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z)) + px + qz + \bar{u}\\ y &= z\end{aligned} \tag{51}$$

The slow state $x$ is not measured. The full-state feedback control law $u$ given by (11) is modified to the output feedback control

$$\bar{u} = -(p + k_1k_2)\hat{x} - (q + k_2)z \tag{52}$$

where $\hat{x}$ is the estimate of $x$. In the fast time-scale, the slow state $x$ stays close to its initial condition $x(0)$, but the initial condition $x(0)$ is not captured from the measurement. The observer needs to ensure that $\hat{x}$ converges to $x$ in the fast time-scale. The observer is designed using the reduced fast subsystem

$$\begin{aligned}x' &= 0\\ z' &= px + qz + \bar{u}\\ y &= z\end{aligned} \tag{53}$$

obtained by substituting $\varepsilon = 0$ in the full-order dynamics (51). Consider a state transformation $\xi := x - lz$, where $l$ is a gain to be chosen later. The estimate of $\xi$ is $\hat{\xi} := \hat{x} - lz$. If the observer produces an estimate $\hat{\xi}$, an estimate of the slow state $x$ can immediately be computed as $\hat{x} = \hat{\xi} + lz$. A Lyapunov function candidate for the observer and its time-derivative for the reduced fast subsystem (53) are

$$\begin{aligned}V_{f_o} &= \tfrac{1}{2}(\xi - \hat{\xi})^2\\ V'_{f_o}|_{(53)} &= (\xi - \hat{\xi})((x'|_{(53)} - lz'|_{(53)} - \hat{\xi}')\\ &= (\xi - \hat{\xi})(-l(p(\xi + lz) + qz + \bar{u}) - \hat{\xi}')\\ &= (\xi - \hat{\xi})(-lp\xi - (l^2 + q)z - l\bar{u} - \hat{\xi}')\end{aligned} \tag{54}$$

Select observer dynamics

$$\hat{\xi}' = -lp\hat{\xi} - (l^2 + q)y - l\bar{u} \tag{55}$$

such that the time-derivative of the observer Lyapunov function for the reduced fast subsystem becomes

$$V'_{f_o}|_{(53)} = -lp(\xi - \hat{\xi})^2 \tag{56}$$

For a nonzero $p$, if the gain $l$ is chosen to be of the same sign as $p$ such that $lp > 0$, the time-derivative (56) is negative-definite, indicating the equilibrium $\hat{\xi} = \xi$ is asymptotically stable. Noting that $\hat{\xi} = \xi$ translates to the equilibrium $\hat{x} = x$, the observer ensures convergence of the slow state estimate $\hat{x}$ to the actual slow state $x$ for the reduced-order dynamics.

An extension of the composite Lyapunov analysis (Khalil, 2002) is performed in order to find the upper bound $\varepsilon^{**}$ of $\varepsilon$, up to which stability of the original full-order nonlinear system (50) is ensured. A Lyapunov function candidate for the full-order system is constructed as

$$V_{c_{ob}} = \beta_1 V_{s_c} + \beta_2 V_{f_c} + \beta_3 V_{f_o}. \tag{57}$$

Similar to the case of slow state measured, this function is a composite of the two individual Lyapunov functions used for controller design and one Lyapunov function used for observer design. This is positive-definite for any $\beta_1, \beta_2, \beta_3 > 0$. The following theorem gives the bound of stability for the full-order nonlinear system (50).

**Theorem 3.3:** *Suppose that the system parameter $p \neq 0$. Suppose that the gains $k_1 > 0, k_2 > 0, \beta_1 > 0, \beta_2 > 0, \beta_3 > 0$, and $l$ satisfying $lp > 0$ can be selected such that the following inequalities hold:*

$$\begin{aligned}&2\beta_1k_1 > |\beta_2k_1^2 - \beta_1| + (\beta_2 + \beta_3|l|)(F_1G_1 + F_2G_2k_1) + \beta_3k_1\\ &2k_2 > |p + k_1k_2|\\ &2\beta_3lp > \beta_2|p + k_1k_2|\end{aligned} \tag{58}$$

*then the output feedback control law (52) supported by the observer (55) keeps the equilibrium $x = 0, z = z^0, \hat{x} = x$ of the full-order nonlinear system (50), (55) globally asymptotically stable, and therefore $\hat{x} \to x$, $z \to z^0$, $x \to 0$ as $t \to \infty$ from any set of initial conditions $\hat{x}(0), z(0), x(0)$ for $0 < \varepsilon < \varepsilon^{**}$ where $\varepsilon^{**} = \min(\varepsilon_1^{**}, \varepsilon_2^{**})$ with*

$$\begin{aligned}\varepsilon_1^{**} &= \frac{\beta_2(2k_2 - |p + k_1k_2|)}{\beta_2(2k_1 + 2F_2G_2 + F_1G_1 + F_2G_2k_1) + |\beta_2k_1^2 - \beta_1| + \beta_3(1 + |l|F_2G_2)}\\ \varepsilon_2^{**} &= \frac{2\beta_3lp - \beta_2|p + k_1k_2|}{\beta_3(1 + k_1 + |l|(F_1G_1 + F_2G_2 + F_2G_2k_1))}\end{aligned} \tag{59}$$

***Proof:*** The proof is similar to that of Theorem 3.2. An outline of the proof is presented. The observer Lyapunov function is $V_{f_o} = \frac{1}{2}(\xi - \hat{\xi})^2 = \frac{1}{2}(x - \hat{x})^2$, and its time-derivative for the reduced fast subsystem is $V'_{f_o}|_{(53)} = -lp(\xi - \hat{\xi})^2 = -lp(x - \hat{x})^2$. These follow from the definitions of $\xi$ and $\hat{\xi}$. Introducing appropriate reduced subsystems, the time-derivative of the composite Lyapunov function can be written as

$$\begin{aligned}\dot{V}_{c_{ob}} &= \beta_1\dot{V}_{s_c}|_{(2)} + \frac{\beta_2}{\varepsilon}V'_{f_c}|_{(8)} + \frac{\beta_3}{\varepsilon}V'_{f_o}|_{(53)}\\ &\quad + \beta_1(\dot{V}_{s_c}|_{(50)} - \dot{V}_{s_c}|_{(2)})\\ &\quad + \frac{\beta_2}{\varepsilon}(V'_{f_c}|_{(51)} - V'_{f_c}|_{(8)}) + \frac{\beta_3}{\varepsilon}(V'_{f_o}|_{(51)} - V'_{f_o}|_{(53)}).\end{aligned} \tag{60}$$

Substituting for the time-derivatives of individual Lyapunov functions,

$$\begin{aligned}\dot{V}_{c_{ob}} &= -\beta_1k_1x^2 - \frac{\beta_2}{\varepsilon}k_2(z - z^0)^2\\ &\quad - \frac{\beta_3}{\varepsilon}lp(x - \hat{x})^2 + \beta_1x(z - z^0)\end{aligned}$$

$$+\frac{\beta_2}{\varepsilon}(z-z^0)[\varepsilon(f_1(x)g_1(z)+f_2(x)g_2(z)) + \bar{u}-u+k_1\varepsilon z] + \frac{\beta_3}{\varepsilon}(x-\hat{x})[\varepsilon z - l\varepsilon(f_1(x)g_1(z)+f_2(x)g_2(z))]. \quad (61)$$

This can be simplified using (20), (21), and the following results:

$$\begin{aligned}
&\bar{u}-u = (p+k_1k_2)(x-\hat{x})\\
&(x-\hat{x})z = (x-\hat{x})(z-z^0) - k_1x(x-\hat{x})\\
&(x-\hat{x})f_1(x)g_1(z) \le F_1G_1|x||x-\hat{x}|\\
&(x-\hat{x})f_2(x)g_2(z) \le F_2G_2|z-z^0||x-\hat{x}| + F_2G_2k_1|x||x-\hat{x}|
\end{aligned} \quad (62)$$

Applying these results and collecting coefficients,

$$\begin{aligned}
\dot{V}_{c_{ob}} \le& -\beta_1k_1x^2 - \beta_2\left(\frac{k_2}{\varepsilon}-k_1-F_2G_2\right)(z-z^0)^2\\
&-\frac{\beta_3}{\varepsilon}lp(x-\hat{x})^2\\
&+(|\beta_2k_1^2-\beta_1| + \beta_2(F_1G_1+F_2G_2k_1))|x||z-z^0|\\
&+(\beta_3(1+|l|F_2G_2) + \frac{\beta_2}{\varepsilon}|p+k_1k_2|)|z-z^0||x-\hat{x}|\\
&+\beta_3(k_1+|l|F_1G_1+|l|F_2G_2k_1)|x||x-\hat{x}|
\end{aligned} \quad (63)$$

Using completion of squares for the product terms $|x||z-z^0|$, $|z-z^0||x-\hat{x}|$, $|x||x-\hat{x}|$, the time-derivative of the composite Lyapunov function can be expressed in the following vector-matrix form:

$$\dot{V}_{c_{ob}} \le -\grave{\mathbb{X}}^{\mathrm{T}}\mathbb{M}\grave{\mathbb{X}} \quad (64)$$

where $\grave{\mathbb{X}} := \begin{bmatrix} x \\ z-z^0 \\ x-\hat{x}\end{bmatrix}$ and $\mathbb{M} := \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33}\end{bmatrix}$, with the diagonal elements $\mu_{11}, \mu_{22}, \mu_{33}$ given by

$$\begin{aligned}
\mu_{11} =& \beta_1k_1 - \frac{1}{2}\left(|\beta_2k_1^2-\beta_1| + \beta_2F_1G_1 + \beta_2F_2G_2k_1 + \beta_3k_1 + \beta_3|l|F_1G_1 + \beta_3|l|F_2G_2k_1\right)\\
\mu_{22} =& \beta_2\left(\frac{k_2}{\varepsilon}-k_1-F_2G_2\right) - \frac{1}{2}\left(|\beta_2k_1^2-\beta_1| + \beta_2F_1G_1 + \beta_2F_2G_2k_1 + \frac{\beta_2}{\varepsilon}|p+k_1k_2| + \beta_3 + \beta_3|l|F_2G_2\right)\\
\mu_{33} =& \frac{\beta_3}{\varepsilon}lp - \frac{1}{2}\left(\beta_3 + \beta_3|l|F_2G_2 + \beta_3k_1 + \beta_3|l|F_1G_1 + \beta_3|l|F_2G_2k_1 + \frac{\beta_2}{\varepsilon}|p+k_1k_2|\right)
\end{aligned} \quad (65)$$

If all the diagonal elements of the matrix $\mathbb{M}$ are positive, the time-derivative of the composite Lyapunov function will be negative-definite everywhere in the state-space, and hence the equilibrium $x = 0, z = z^0, \hat{x} = x$ of the full-order nonlinear system will be globally asymptotically stable. The element $\mu_{11}$ is positive if the first sufficient condition in (58) holds. The element $\mu_{22}$ is positive if $0 < \varepsilon < \varepsilon_1^{**}$, where $\varepsilon_1^{**}$ is given by (59). The upper bound $\varepsilon_1^{**}$ is positive if the second sufficient condition in (58) holds. The element $\mu_{33}$ is positive if $0 < \varepsilon < \varepsilon_2^{**}$ where $\varepsilon_2^{**}$ is given by (59). The upper bound $\varepsilon_2^{**}$ is positive if the third sufficient condition in (58) holds. Therefore, if all the three sufficient conditions in (58) hold, global asymptotic stability holds for $0 < \varepsilon < \varepsilon^{**}$ where $\varepsilon^{**} = \min(\varepsilon_1^{**}, \varepsilon_2^{**})$. This completes the proof. ■

### 3.2.3 Linear combination of slow and fast states measured

For this case, the full-order system in the slow time-scale is

$$\begin{aligned}
\dot{x} &= z\\
\varepsilon\dot{z} &= \varepsilon(f_1(x)g_1(z)+f_2(x)g_2(z)) + px + qz + \bar{u}\\
y &= c_1x + c_2z.
\end{aligned} \quad (66)$$

In the fast time-scale, the full-order system is

$$\begin{aligned}
x' &= \varepsilon z\\
z' &= \varepsilon(f_1(x)g_1(z)+f_2(x)g_2(z)) + (px+qz) + \bar{u}\\
y &= c_1x + c_2z.
\end{aligned} \quad (67)$$

Since none of the states $x$ and $z$ are directly measured, the control law is modified from its full-state version (11) to

$$\bar{u} = -(p+k_1k_2)\hat{x} - (q+k_2)\hat{z} \quad (68)$$

where $\hat{x}$ and $\hat{z}$ are the estimates of the states. An observer needs to estimate both $x$ and $z$ in the fast time-scale. The observer is designed using the reduced fast subsystem

$$\begin{aligned}
x' &= 0\\
z' &= px + qz + \bar{u}\\
y &= c_1x + c_2z.
\end{aligned} \quad (69)$$

Assume the observer dynamics to be functions of the estimated states $\hat{x}, \hat{z}$, control $\bar{u}$ and output $y$ as

$$\begin{aligned}
\hat{x}' &= \phi_1(\hat{x}, \hat{z}, \bar{u}, y)\\
\hat{z}' &= \phi_2(\hat{x}, \hat{z}, \bar{u}, y)
\end{aligned} \quad (70)$$

where the functions $\phi_1(.)$ and $\phi_2(.)$ are to be chosen such that the estimated states $\hat{x}, \hat{z}$ converge to the actual states $x, z$ in the fast time-scale. A positive-definite Lyapunov function candidate for the observer is

$$V_{f_o} = \tfrac{1}{2}\gamma_1(x-\hat{x})^2 + \tfrac{1}{2}\gamma_2(z-\hat{z})^2 \quad (71)$$

where $\gamma_1, \gamma_2 > 0$. For the reduced fast subsystem (69), the time-derivative of this function is

$$\begin{aligned}
V'_{f_o}|_{(69)} &= \gamma_1(x-\hat{x})(x'|_{(69)} - \hat{x}'|_{(69)}) + \gamma_2(z-\hat{z})(z'|_{(69)} - \hat{z}'|_{(69)})\\
&= -\gamma_1(x-\hat{x})\phi_1(.) + \gamma_2(z-\hat{z})(px+qz+\bar{u}-\phi_2(.)).
\end{aligned} \quad (72)$$

Select the fast observer dynamics as

$$\begin{aligned}\phi_1(.) &= l_1(y - c_1\hat{x} - c_2\hat{z})\\ \phi_2(.) &= p\hat{x} + q\hat{z} + \bar{u} + l_2(y - c_1\hat{x} - c_2\hat{z})\end{aligned} \tag{73}$$

where $l_1, l_2$ are observer gains. By this choice the time derivative of the observer Lyapunov function becomes

$$V'_{f_o}|_{(69)} = -\breve{\mathbb{X}}^{\mathrm{T}} Q \breve{\mathbb{X}} \tag{74}$$

where $\breve{\mathbb{X}} := [x - \hat{x} \quad z - \hat{z}]^{\mathrm{T}}$ and

$$Q := \begin{bmatrix} \gamma_1 l_1 c_1 & \frac{1}{2}(\gamma_1 l_1 c_2 + \gamma_2 l_2 c_1 - \gamma_2 p) \\ \frac{1}{2}(\gamma_1 l_1 c_2 + \gamma_2 l_2 c_1 - \gamma_2 p) & \gamma_2(l_2 c_2 - q) \end{bmatrix} \tag{75}$$

If the gains $\gamma_1, \gamma_2, l_1, l_2$ are selected such that

$$\begin{aligned}&\gamma_1 l_1 c_1 > 0\\ &\gamma_1 l_1 c_1 \gamma_2(l_2 c_2 - q) > \tfrac{1}{4}(\gamma_1 l_1 c_2 + \gamma_2 l_2 c_1 - \gamma_2 p)^2\end{aligned} \tag{76}$$

then Q is positive-definite, and consequently, the time-derivative of the Lyapunov function $V_{f_o}$ for the reduced fast subsystem is negative-definite. It can be said that

$$\begin{aligned}V'_{f_o}|_{(69)} &= -\breve{\mathbb{X}}^{\mathrm{T}} Q \breve{\mathbb{X}} \leq -\lambda_{\min}(Q)\breve{\mathbb{X}}^{\mathrm{T}}\breve{\mathbb{X}}\\ &= -\lambda((x-\hat{x})^2 + (z-\hat{z})^2)\end{aligned} \tag{77}$$

where $\lambda := \lambda_{\min}(Q)$ is the minimum eigenvalue of the matrix Q. Since Q is positive-definite, $\lambda > 0$, and its magnitude is dictated by the choices of $\gamma_1, \gamma_2, l_1, l_2$. Equation (77) indicates that the estimates $\hat{x}, \hat{z}$ will converge to the true states $x, z$ for the reduced fast subsystem (69).

Similar to the previous two cases of output feedback, an extension of the composite Lyapunov analysis is performed to find out an upper bound $\varepsilon^{**}$ of the perturbation parameter $\varepsilon$ such that the full-order nonlinear system with the controller and the observer in the loop is Lyapunov-stable. A candidate composite Lyapunov function for the full-order system (66) is

$$V_{c_{ob}} = \delta_1 V_{s_c} + \delta_2 V_{f_c} + \delta_3 V_{f_o} \tag{78}$$

This function is positive-definite and radially unbounded for any $\delta_1, \delta_2, \delta_3 > 0$. The following theorem gives the bound of $\varepsilon$ for the full-order system (66) to be globally asymptotically stable.

**Theorem 3.4:** *Suppose that the gains $k_1 > 0, k_2 > 0, \delta_1 > 0, \delta_2 > 0, \delta_3 > 0$ and $\gamma_1 > 0, \gamma_2 > 0, l_1, l_2$ satisfying (76) can be selected such that the following inequalities hold:*

$$\begin{aligned}&2\delta_1 k_1 > |\delta_2 k_1^2 - \delta_1| + \delta_2(F_1G_1 + F_2G_2k_1)\\ &\quad + \delta_3(\gamma_1 k_1 + \gamma_2 F_1 G_1 + \gamma_2 F_2 G_2 k_1)\\ &2k_2 > |p + k_1k_2| + |q + k_2|\\ &2\delta_3\lambda > \delta_2|p + k_1k_2|\\ &2\delta_3\lambda > \delta_2|q + k_2|\end{aligned} \tag{79}$$

*where $\lambda$ is the minimum eigenvalue of the matrix Q defined by (75). Then the output feedback controller (68) supported by the observer (73) ensures that the equilibrium $x = 0, z = z^0, \hat{x} = x, \hat{z} = z$ of the full-order system (66), (73) is globally asymptotically stable, and therefore $\hat{z} \to z, \hat{x} \to x, z \to z^0, x \to 0$ as $t \to \infty$ from any set of initial conditions $\hat{z}(0), \hat{x}(0), z(0), x(0)$ for $0 < \varepsilon < \varepsilon^{**}$ where $\varepsilon^{**} = \min(\varepsilon_1^{**}, \varepsilon_2^{**}, \varepsilon_3^{**})$ with*

$$\begin{aligned}\varepsilon_1^{**} &= \frac{\delta_2(2k_2 - |p + k_1k_2| - |q + k_2|}{\delta_2(F_1G_1 + F_2G_2(2 + k_1) + 2k_1)\\ \quad + |\delta_2 k_1^2 - \delta_1| + \delta_3(\gamma_1 + \gamma_2 F_2 G_2}\\ \varepsilon_2^{**} &= \frac{2\delta_3\lambda - \delta_2|p + k_1k_2|}{\delta_3(1 + k_1)\gamma_1}\\ \varepsilon_3^{**} &= \frac{2\delta_3\lambda - \delta_2|q + k_2|}{\delta_3\gamma_2(F_1G_1 + F_2G_2(1 + k_1))}\end{aligned} \tag{80}$$

***Proof:*** The proof of this Theorem is similar to the ones of Theorems 3.1 and 3.2. An outline of the proof is presented. Adding and subtracting the time-derivatives of the Lyapunov functions for the appropriate reduced subsystems,

$$\begin{aligned}\dot{V}_{c_{ob}} &= \delta_1 \dot{V}_{s_c}|_{(2)} + \frac{\delta_2}{\varepsilon}\bigg|_{(8)} + \frac{\delta_3}{\varepsilon}\bigg|_{(69)} + \delta_1(\dot{V}_{s_c}|_{(66)} - \dot{V}_{s_c}|_{(2)})\\ &\quad + \frac{\delta_2}{\varepsilon}(V'_{f_c}|_{(67)} - V'_{f_c}|_{(8)}) + \frac{\delta_3}{\varepsilon}(V'_{f_o}|_{(67)} - V'_{f_o}|_{(69)})\end{aligned} \tag{81}$$

Substituting for the Lyapunov functions and their time-derivatives,

$$\begin{aligned}\dot{V}_{c_{ob}} &\leq -\delta_1 k_1 x^2 - \frac{\delta_2}{\varepsilon}k_2(z - z^0)^2 - \frac{\delta_3}{\varepsilon}\lambda(x - \hat{x})^2\\ &\quad - \frac{\delta_3}{\varepsilon}\lambda(z - \hat{z})^2 + \delta_1 x(z - z^0)\\ &\quad + \frac{\delta_2}{\varepsilon}(z - z^0)[\varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z))\\ &\quad + \bar{u} - u + k_1\varepsilon z]\\ &\quad + \frac{\delta_3}{\varepsilon}\gamma_1(x - \hat{x})\varepsilon z\\ &\quad + \frac{\delta_3}{\varepsilon}\gamma_2(z - \hat{z})\varepsilon(f_1(x)g_1(z) + f_2(x)g_2(z))\end{aligned} \tag{82}$$

Expressing the difference between the output feedback control $\bar{u}$ and full-state feedback control $u$ as $\bar{u} - u = (p + k_1k_2)(x - \hat{x}) + (q + k_2)(z - \hat{z})$, using the bounds (20), (21), (43), (44), (62) for the terms involving the nonlinear functions $f_1(x), g_1(z), f_2(x), g_2(z)$, and performing completion of squares for product terms like $|x||z - z^0|$, $|x - \hat{x}||z - z^0|$, etc., the time-derivative of the composite Lyapunov function becomes the following inequality:

$$\dot{V}_{c_{ob}} \leq -\tilde{\mathbb{X}}^{\mathrm{T}}\mathbb{E}\tilde{\mathbb{X}} \tag{83}$$

where $\tilde{\mathbb{X}} = \begin{bmatrix} x \\ z-z^0 \\ x-\hat{x} \\ z-\hat{z} \end{bmatrix}$ and $\mathbb{E} = \begin{bmatrix} \eta_{11} & 0 & 0 & 0 \\ 0 & \eta_{22} & 0 & 0 \\ 0 & 0 & \eta_{33} & 0 \\ 0 & 0 & 0 & \eta_{44} \end{bmatrix}$ with the diagonal elements $\eta_{ii}$; $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned}
\eta_{11} &:= \delta_1 k_1 - \frac{1}{2}(|\delta_2 k_1^2 - \delta_1| \\
&\quad + (\delta_2 + \delta_3\gamma_2)(F_1G_1 + F_2G_2k_1) + \delta_3\gamma_1 k_1) \\
\eta_{22} &:= \delta_2\left(\frac{k_2}{\varepsilon} - k_1 - F_2G_2\right) \\
&\quad - \frac{1}{2}\left(|\delta_2 k_1^2 - \delta_1| + \frac{\delta_2}{\varepsilon}|p + k_1k_2| + \frac{\delta_2}{\varepsilon}|q + k_2|\right. \\
&\quad \left. + \delta_2(F_1 + F_2G_2k_1) + \delta_3(\gamma_1 + \gamma_2F_2G_2)\right) \\
\eta_{33} &:= \frac{\delta_3}{\varepsilon}\lambda - \frac{1}{2}\left(\frac{\delta_2}{\varepsilon}|p + k_1k_2| + \delta_3(1 + k_1)\gamma_1\right) \\
\eta_{44} &:= \frac{\delta_3}{\varepsilon}\lambda - \frac{1}{2}\left(\frac{\delta_2}{\varepsilon}|q + k_2| + \delta_3\gamma_2(F_1G_1 + F_2G_2(1 + k_1))\right)
\end{aligned} \tag{84}$$

If all the diagonal elements of the matrix $\mathbb{E}$ are positive, the time-derivative of the composite Lyapunov function is negative-definite everywhere in the state-space, and therefore the full-order system is globally asymptotically stable. The element $\eta_{11}$ is positive if the first sufficient condition in (79) is met. The element $\eta_{22} > 0$ if $0 < \varepsilon < \varepsilon_1^{**}$, where the upper bound $\varepsilon_1^{**}$ is specified in (80). The second sufficient condition in (79) ensures that $\varepsilon_1^{**}$ is positive. The element $\eta_{33} > 0$ if $0 < \varepsilon < \varepsilon_2^{**}$, where $\varepsilon_2^{**}$ is specified in (80). The third sufficient condition in (79) ensures that $\varepsilon_2^{**}$ is positive. The element $\eta_{44} > 0$ if $0 < \varepsilon < \varepsilon_3^{**}$, where $\varepsilon_3^{**}$ is specified in (80). The fourth sufficient condition in (79) ensures that $\varepsilon_3^{**}$ is positive. Therefore, if all four of the sufficient conditions in (79) are satisfied, global asymptotic stability is guaranteed for $0 < \varepsilon < \varepsilon^{**} = \min(\varepsilon_1^{**}, \varepsilon_2^{**}, \varepsilon_3^{**})$. This completes the proof. ■

## 4. Numerical examples

This section compares in simulation the performances of the full-state feedback controller and the three different cases of output feedback controllers developed in Section 3. In addition, by choosing suitable gains for all of the cases, bounds of time-scale separation are numerically evaluated such that stability of the full-order nonlinear system is guaranteed within those bounds.

For simulation, the time-scale separation parameter $\varepsilon$ is assumed 0.01. The initial conditions of the states are $x(0) = 5, z(0) = 5$. The system parameters $p$ and $q$ are assumed as $p = 0.1$, $q = -0.9$. The nonlinear functions are assumed as follows:

$$\begin{aligned}
f_1(x) &= F_1 x \sin x \\
g_1(z) &= \begin{cases} -G_1 & z < -0.1 \\ 10G_1 z & -0.1 \le z < 0.1 \\ G_1 & z \ge 0.1 \end{cases} \\
f_2(x) &= F_2 \cos x \\
g_2(z) &= \tfrac{2}{243}\mathrm{e}^3 G_2 z^3\, \mathrm{e}^{-\sqrt{|z|}}.
\end{aligned} \tag{85}$$

The sector-bounds are $F_1 = 1, G_2 = 1$, and the magnitude-bounds are $F_2 = 0.1, G_1 = 0.1$. As mentioned in Section 2, for control law development only the bounds $F_1, G_1, F_2, G_2$ are used. The exact expressions of the nonlinear functions are used only to simulate the dynamics.

For full-state feedback, the controller gains are selected as $k_1 = 1, k_2 = 1$. Figures 1(a) and 2(a) show the states and control for full-state feedback. It can be seen in Figure 1(a) that the slow state remains almost constant at its initial condition, while the fast state converges to its manifold in the fast time-scale. Figure 2(a) show that the slow state converges to zero in the slow time-scale, while the fast state stays on its manifold. For output feedback with slow state measured, the control gains are selected as $k_1 = 1, k_2 = 2$. Figures 1(b) and 2(b) show the time-histories of the states and the control in the fast and slow time-scales respectively for the case of only the slow state being measured. For this case the observer produces estimates of the fast state. The initial estimate of the fast state is $\hat{z}(0) = 0$. Figure 1(b) shows that in the fast time-scale the estimate of the fast state converges to the actual fast state, and that the fast state converges to its manifold. Figure 2(b) shows that the slow state $x$ goes to zero in the slow time-scale, and that the fast state stays on its manifold.

Figures 3(b) and 4(b) show the states and control when the fast state is measured, and the observer produces estimates of the slow state. For this case, the initial estimate of the slow state is $\hat{x}(0) = -5$, the controller gains are $k_1 = 1, k_2 = 2$, and the observer gain is $l = 10$. Figure 3(b) shows the convergence of the estimated slow state to the actual slow state, and that of the fast state to its manifold in the fast time-scale. Subsequently, the slow state reaches the origin while the fast state stays on its manifold, as seen in Figure 4(b). For the case of output feedback with a linear combination of slow and fast states measured, the observer produces estimates of both of the slow and the fast states. The numbers $c_1, c_2$ in the output equations are assumed as $c_1 = 0.1, c_2 = 1$. The initial estimates are $\hat{x}(0) = -5, \hat{z}(0) = 0$. The observer gains are selected as $l_1 = 10, l_2 = 5$, and the controller gains are chosen as $k_1 = 1, k_2 = 1$. Figure 5(b) shows that both of the state estimates converge to the actual states in the fast time-scale, and at the same time the fast state converges to its manifold. Figure 6(b) that the slow state converges to origin, and the fast state stays on its manifold in the slow time-scale. In essence, for all of three cases of output feedback, the state trajectories are similar to the ones for full-state feedback once the observed states converge to the actual states in the fast time-scale.

Table 1 shows a numerical comparison of the bounds of $\varepsilon$ for full-state and output feedback. Under output feedback there are a total of three cases: slow state measured, fast state measured, and linear combination of states measured. Consequently Table 1 has one column for full-state feedback, and three columns for the three different cases of output feedback. The first few rows of Table 1 list the system parameters, known bounds of nonlinear functions, and the initial conditions of the states. These are the same for every case. The subsequent row lists the initial estimates of the observed states. For full-state feedback, there is no estimated state; however, for output feedback there are either one or two estimated states. The next few rows of Table 1 list the gains: there are some gains used in

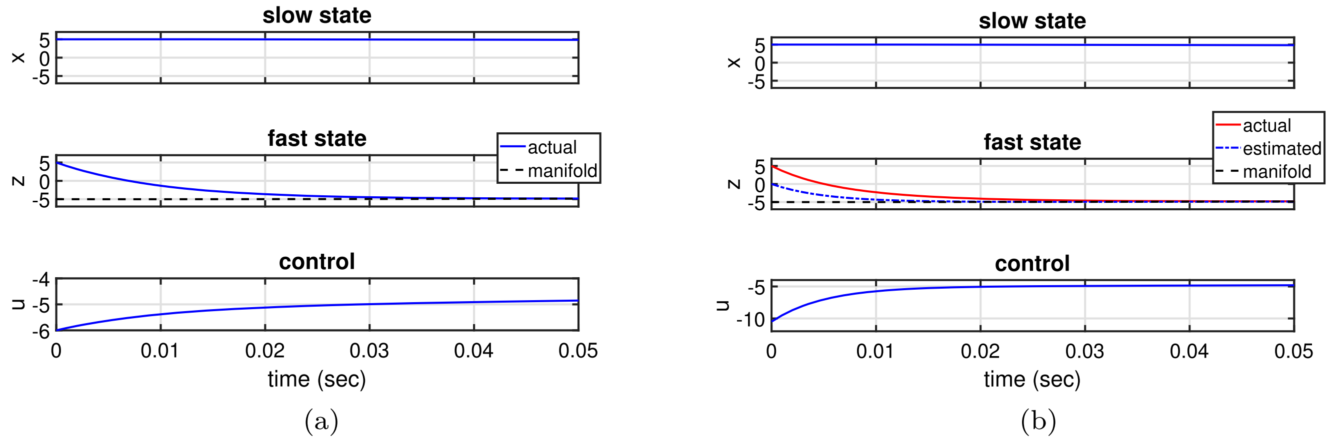


**Figure 1.** Comparison in the fast time-scale between full-state feedback and output feedback with slow state measured. (a) full-state feedback. (b) output feedback, slow state measured.

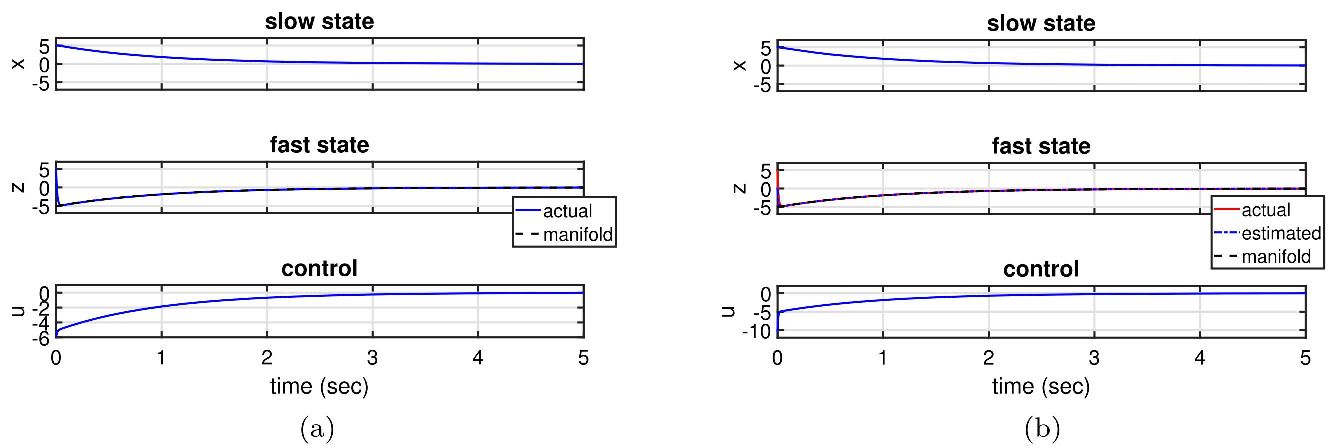


**Figure 2.** Comparison in the slow time-scale between full-state feedback and output feedback with slow state measured. (a) full-state feedback. (b) output feedback, slow state measured.

the control law, some in the observer dynamics, and some to construct the composite Lyapunov functions. These gains are different from one case to another so they satisfy the sufficient conditions for the corresponding cases. For example, for the case of fast state measured, the gains are chosen so they satisfy the constraints in Theorem 3.2. The following row of Table 1 shows the candidate bounds of time-scale separation for each case. These candidate bounds are computed according to Theorems 3.1–3.4. For the case of full-state feedback, there is only one candidate bound $\varepsilon^*$, but for each case of output feedback there are multiple candidate bounds of $\varepsilon$. The $\varepsilon^{**}$ reported in the last row is the minimum of those candidate bounds.

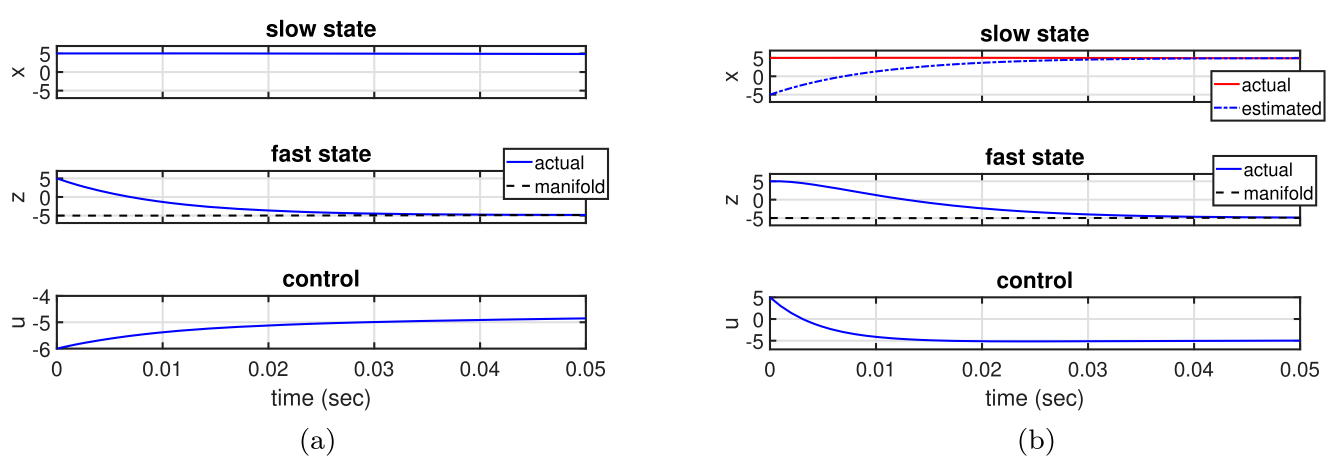


**Figure 3.** Comparison in the fast time-scale between full-state feedback and output feedback with fast state measured. (a) full-state feedback. (b) output feedback, fast state measured.

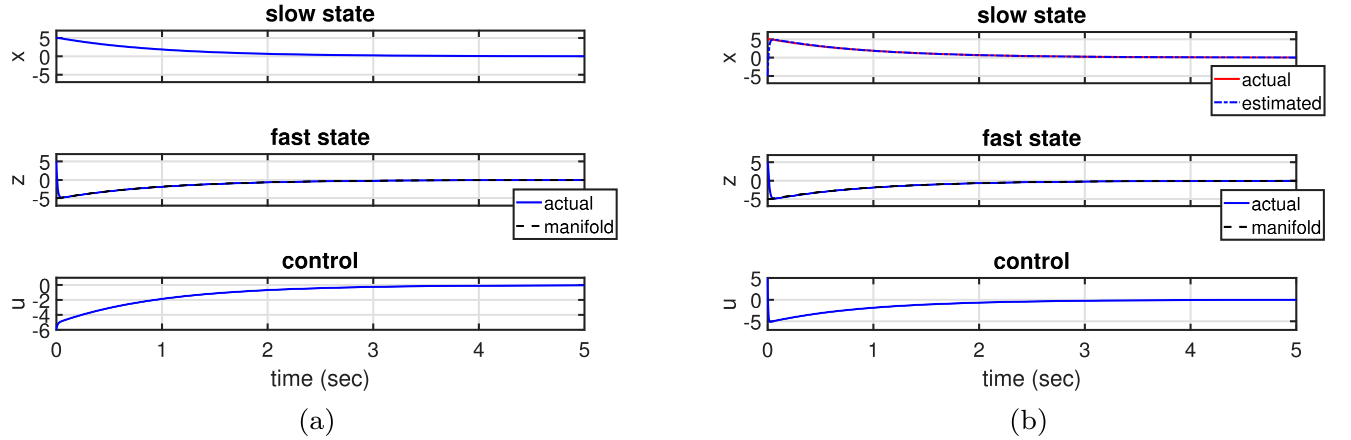


**Figure 4.** Comparison in the slow time-scale between full-state feedback and output feedback with fast state measured. (a) full-state feedback. (b) output feedback, fast state measured.

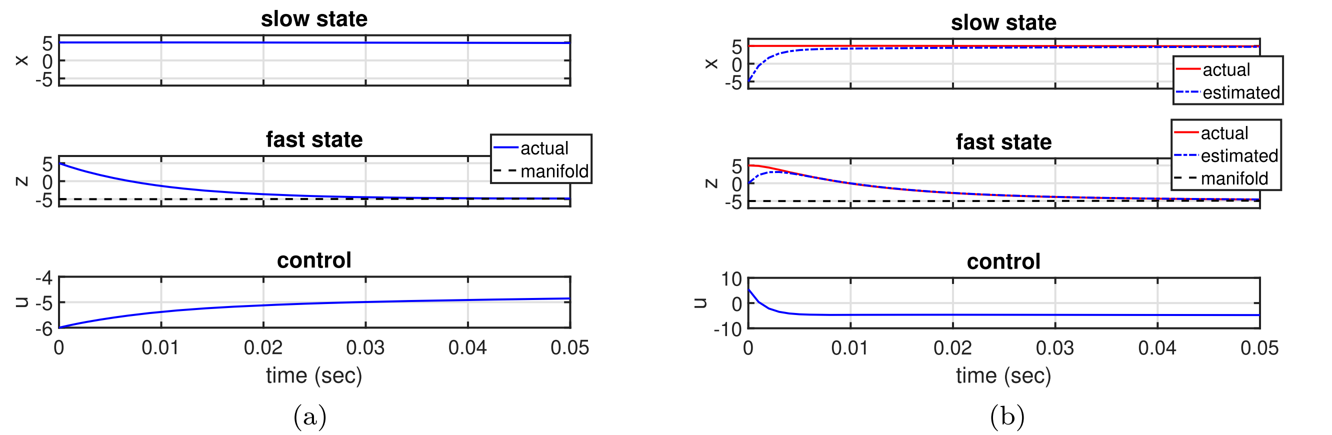


**Figure 5.** Comparison in the fast time-scale between full-state feedback and output feedback with combination of slow and fast states measured. (a) full-state feedback. (b) output feedback, combination of states measured.

A way to interpret the final bounds $\varepsilon^*$ and $\varepsilon^{**}$ in the last row of Table 1 is as follows. For full-state feedback the current simulation yields $\varepsilon^* = 0.9$. This indicates that the full-state feedback controller can ensure stability when the evolution of the fast state is faster than $1/0.9 \approx 1.1$ times that of the slow state. Similarly, the output feedback controller with slow state measured can ensure stability when the evolution of the fast state is faster than $1/0.6 \approx 1.7$ times that of the slow state. For the other two cases of output feedback, stability is guaranteed if the fast state evolves faster than $1/0.15 \approx 6.7$ or $1/0.03 \approx 33$ times that of the slow state.

**Remark 4.1:** These numbers representing the bounds of stability can be altered by selecting different (a) controller gains,

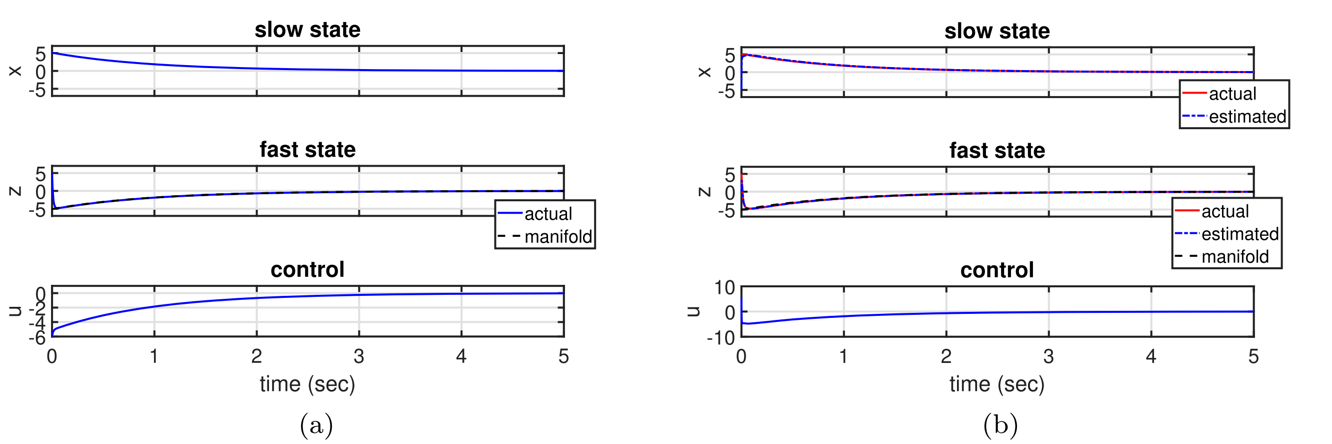


**Figure 6.** Comparison in the slow time-scale between full-state feedback and output feedback with combination of slow and fast states measured. (a) full-state feedback and (b) output feedback, combination of states measured.

**Table 1.** Comparison of stability bounds.

| | Full-state feedback | Output feedback $y = x$ | Output feedback $y = z$ | Output feedback $y = c_1x + c_2z$ $c_1 = 0.1$ $c_2 = 1$ |
|---|---|---|---|---|
| System parameters | $p = 0.1$ $q = -0.9$ | $p = 0.1$ $q = -0.9$ | $p = 0.1$ $q = -0.9$ | $p = 0.1$ $q = -0.9$ |
| Sector-bounds and magnitude-bounds of nonlinearities | $F_1 = 1$ $G_1 = 0.1$ $F_2 = 0.1$ $G_2 = 1$ | $F_1 = 1$ $G_1 = 0.1$ $F_2 = 0.1$ $G_2 = 1$ | $F_1 = 1$ $G_1 = 0.1$ $F_2 = 0.1$ $G_2 = 1$ | $F_1 = 1$ $G_1 = 0.1$ $F_2 = 0.1$ $G_2 = 1$ |
| Initial conditions of the states | $x(0) = 5$ $z(0) = 5$ | $x(0) = 5$ $z(0) = 5$ | $x(0) = 5$ $z(0) = 5$ | $x(0) = 5$ $z(0) = 5$ |
| Initial estimates of observed states | none | $\hat{z}(0) = 0$ | $\hat{x}(0) = -5$ | $\hat{x}(0) = -5$ $\hat{z}(0) = 0$ |
| Controller gains | $k_1 = 1$ $k_2 = 1$ | $k_1 = 1$ $k_2 = 2$ | $k_1 = 1$ $k_2 = 2$ | $k_1 = 1$ $k_2 = 1$ |
| Observer gains | none | no separate observer gain | $l = 10$ | $l_1 = 10$ $l_2 = 5$ |
| Gains corresponding to weights of individual Lyapunov functions | $w_1 = 1$ $w_2 = 1$ | $\alpha_1 = 1$ $\alpha_2 = 0.3$ $\alpha_3 = 0.3$ | $\beta_1 = 0.75$ $\beta_2 = 0.1$ $\beta_3 = 0.2$ | $\gamma_1 = 1$ $\gamma_2 = 10$ $\delta_1 = 0.75$ $\delta_2 = 0.05$ $\delta_3 = 0.25$ |
| Candidate bounds of $\varepsilon$ | $\varepsilon^* = 0.9$ | $\varepsilon_1^{**} = 0.6$ $\varepsilon_2^{**} = 2.3$ | $\varepsilon_1^{**} = 0.15$ $\varepsilon_2^{**} = 0.19$ | $\varepsilon_1^{**} = 0.03$ $\varepsilon_2^{**} = 0.06$ $\varepsilon_3^{**} = 0.1$ |
| Final bound of $\varepsilon$ | $\varepsilon^* = 0.9$ | $\varepsilon^{**} = 0.6$ | $\varepsilon^{**} = 0.15$ | $\varepsilon^{**} = 0.03$ |

(b) observer gains, (c) gains corresponding to weights of individual Lyapunov functions in the composite. With the exception of full-state feedback (Theorem 3.1), for any of the cases of output feedback the new set of gains should satisfy the inequality constraints given in Theorems 3.2, 3.3 or 3.4 so stability is still guaranteed. The gains reported in Table 1 satisfy all the relevant inequalities, but those gains were found using trial and error. The selection of gains can be sequenced as follows. The controller and observer gains typically indicate the closed-loop speeds of response of the errors, so they are selected first. The weights of Lyapunov functions are additional 'tuning knob's which are selected later. However, an analytical computation of a set of gains which will always satisfy these constraints is an open problem.

**Remark 4.2:** If for a system $\varepsilon > \varepsilon^*$ or $\varepsilon > \varepsilon^{**}$, or in other words if the fast state evolves slower than the rates corresponding to $\varepsilon^*$ or $\varepsilon^{**}$, it does not automatically mean that the closed-loop system will be unstable. However, stability of the closed-loop system with the two-time-scale controller cannot be guaranteed without further analysis. If $\varepsilon$ is equal or close to unity, it means that practically the system is not singularly perturbed, i.e. there is no separation of time-scales between the slow and the fast dynamics. In that case a two-time-scale controller may not be needed.

## 5. Conclusions

This paper investigated and developed a theory of output feedback control for a class of nonlinear nonstandard two-time-scale systems. This was achieved by using a sequential controller and a state observer. The controller is designed partly in the slow and partly in the fast time-scale, and the observer is designed entirely in the fast time-scale. Depending upon the measurement, the observer estimates either one or both of the states. Based on the results presented in the paper, the following conclusions can be drawn.

The reduced subsystems being inherently linear simplifies the selections of the controller and observer structures. However, the stability analysis addresses the nonlinearity and shows that the selections of controller and observer gains are dictated by the bounds of the nonlinear functions. This analysis proves that global asymptotic stability for the full-order system with the controller and the observer is guaranteed up to a certain bound of time-scale separation. This bound is obtained in the closed form for all three cases of measurement, and it can be altered by varying the controller gains, observer gains and the weights of individual Lyapunov functions in the composite. The stability proof for each case of output feedback uses completion of squares to find upper bounds of several product terms. This leads to diagonal matrices so only the diagonal entries need to be examined to establish closed-form bounds of time-scale separation as well as constraints on the gains. This is a candidate approach for higher dimensional systems. Slow state regulation is achieved, and the observed states converge to the actual states.

## Disclosure statement

No potential conflict of interest was reported by the authors.

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