Kinetic State Tracking for a Class of Singularly Perturbed Systems

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The trajectory-following control problem for a general class of nonlinear multi-input/multi-output two time-scale system is revisited. While most earlier works used singular perturbation theory and assumed that an isolated real root exists for the nonlinear set of algebraic equations that constitute the slow subsystem, here, two time-scale systems are analyzed in the context of integral manifolds. It is shown that the singularly perturbed system has a center manifold and, for small values of the slow state, an approximate solution of the nonlinear set of transcendental equations can be computed. Geometric singular perturbation theory is used as the model-reduction technique, and modified composite control design is used to formulate the stabilizing control laws for slow state tracking. The control laws are independent of the scalar perturbation parameter and an upper bound for it, and the closed-loop error signals are determined such that uniform boundedness of the closed-loop system is guaranteed. Additionally, asymptotic stabilization is shown for the nonlinear regulation problem. The methodology is demonstrated through numerical simulation of a nonlinear generic two-degree-of-freedom kinetic model and a nonlinear, coupled, six-degree-of-freedom model of the F/A-18A Hornet. Results demonstrate that the methodology permits close tracking of a reference trajectory while maintaining all control signals within specified bounds.

Nomenclature

A, A_f	=	positive gain matrices
b	=	wingspan, ft
c	=	mean aerodynamic chord, ft
C_D	=	drag coefficient
C_L	=	lift coefficient
C_Y	=	side force coefficient
C_l, C_m, C_n	=	roll, pitch, and yaw moment coefficients
D	=	domain of subscripted variable
g	=	gravity acceleration, ft/s^2
I_x, I_y, I_z	=	principal axis inertias for aircraft, slug ft^2
M	=	Mach number
m	=	mass, slug
m	=	number of slow variables
$(M\phi)$	=	nonlinear map
$\mathcal{M}_\epsilon$	=	invariant manifold of full-order system
$\mathcal{M}_0$	=	invariant manifold of reduced-order subsystems
n	=	number of fast variables
$\mathcal{O}(\ )$	=	order symbol

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p	=	number of control variables
p, q, r	=	body roll, pitch, and yaw rates, deg /s
r	=	degree of smoothness
S	=	reference area, ft^2
t	=	slow time scale
T_m	=	maximum thrust, lb
t_0	=	initial time
t^*	=	some finite time; greater than t_0
$\mathbf{u}$	=	control vector
$V(t, \tilde{\mathbf{x}})$	=	Lyapunov function for closed-loop reduced slow subsystem
v_s	=	speed of sound, ft/s
$\mathbf{w}$	=	vector $[\mathbf{x}, \epsilon]^T$
$W(t, \tilde{\mathbf{z}})$	=	Lyapunov function for closed-loop reduced fast subsystem
$\mathbf{x}, \mathbf{z}$	=	state variables of full-order system
$\tilde{\mathbf{x}}$	=	tracking error
$\tilde{\mathbf{z}}$	=	error between fast variable and exact manifold
α	=	angle of attack, deg
β	=	sideslip angle, deg
$\Delta \mathbf{w}, \Delta \mathbf{z}$	=	perturbation quantities
$\delta_e, \delta_a, \delta_r$	=	elevator, aileron, and rudder control inputs, deg
ϵ	=	scalar perturbation parameter
$\epsilon^*(\epsilon_s^*)$	=	upper bound for scalar perturbation parameter (for stabilization problem)
η	=	throttle input
θ	=	pitch attitude angle, deg
μ, γ	=	wind-axes orientation angles, deg
$v(t, \tilde{\mathbf{x}}, \tilde{\mathbf{z}})$	=	Lyapunov function for complete system
ρ	=	density of air, slug/ft^3
τ	=	fast time scale
$\Phi(.)$	=	approximation of exact manifold
ϕ	=	roll attitude angle, deg
$\phi(.)$	=	approximation to exact manifold
ψ	=	heading angle, deg
$\dot{}$	=	derivative with respect to slow time scale
$'$	=	derivative with respect to fast time scale
$\|.\|$	=	Euclidean norm

Subscripts

b	=	bound on variable
r	=	reference

Superscripts

S	=	stable
U	=	unstable

I. Introduction

MATHEMATICAL modeling of many physical systems requires high-order dynamic equations. The presence of parameters such as spring constant, mass, and moments of inertia are the cause of stiffness and increased order of these equations. It is difficult to arrive at exact analytical solutions of these nonlinear governing equations with known, and sometimes unknown, variable coefficients, so an approximate solution is often computed. Singular perturbation theory is a scheme used to simplify systems that inherently possess both fast and slow dynamics. Such systems are characterized by a small parameter ϵ multiplying the highest derivative. Suppression of this small parameter reduces the order of the system, and thus the label of singularly perturbed. Singular perturbation theory dates back to the 1904 work of Prandtl [1] on fluid boundary layers; subsequently, applications of perturbation methods were explored for control design [2–4].

The main contribution of perturbation methods is at the level of modeling, where it has been used as a model-reduction technique as well as a means of removing the numerical stiffness in the original system. In particular, the method of matched asymptotic expansions reduces the study of the full-order system of equations to the study of

two other degenerate models. The first model captures the dominant phenomena, and the neglected phenomena is handled in the second. For the full-order system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \epsilon) \qquad \epsilon \dot{\mathbf{z}} = l(\mathbf{x}, \mathbf{z}, \epsilon) \qquad (1)$$

the lower-order models are developed to be the following:
Reduced slow subsystem,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \epsilon) \qquad 0 = l(\mathbf{x}, \mathbf{z}, \epsilon) \qquad (2)$$

Reduced fast subsystem,

$$\mathbf{x}' = \mathbf{0} \qquad \mathbf{z}' = l(\mathbf{x}, \mathbf{z}, \epsilon) \qquad (3)$$

where ϵ represents the scalar perturbation parameter, and $'$ represents the derivative with respect to the fast time scale $\tau = (t - t_0)/\epsilon$. It has been shown that the behavior of the complete system of Eqs. (1) is constrained within the $\mathcal{O}(\epsilon)$ bound of the reduced slow subsystem, provided the dynamics of the reduced fast subsystem are stabilizing [5]. One problem evident with the reduced slow subsystem is the solution of the transcendental or algebraic set of equations for the fast states $\mathbf{z}$. It is known that there may be many solutions satisfying this set of equations. The standard singular perturbation model assumes that, in the domain of interest, these solutions be isolated real roots.

Tracking properties of standard singularly perturbed systems were first studied by Grujic [6] in 1982. This work laid the foundations of tracking theory in a Lyapunov sense. Later, in 1988, this work was extended for nonlinear time-varying singularly perturbed systems [7]. However, it is assumed that separate controls are available for both the reduced slow and the reduced fast subsystems, and the algebraic set of equations have a trivial solution. Christofides et al. [8] developed robust controller designs for systems with a stabilizable fast subsystem, and input/output linearizable slow subsystems with input-to-state stable inverse dynamics. This work considered a general class of nonlinear time-varying singularly perturbed systems that have dynamics linear in the fast states. Another approach to tracking was presented by Heck [9] in 1991. He addressed the design of sliding-mode controllers for a class of linear time-invariant systems where tracking of slow variables is desired. For both reduced subsystems, a sliding-mode controller is designed, and a composite of these controls is then implemented on the full-order system. The concept of composite control, or designing separate controllers, for each of the subsystems and then implementing their cumulative to the full higher-order system was initiated by Suzuki and Miura [10], and since then, this concept has been extensively used by researchers for robust stabilization of systems with time-scale properties [11–13].

In the aircraft literature, the rotational equations of motion constitute the fast subsystem. These equations are highly coupled and nonlinear; thus, there exist multiple solutions for the set of nonlinear algebraic equations. Tracking of slow variables for these systems is achieved by making two key assumptions. First, the control surface deflections do not affect the slow states. Second, the fast variables are the actuators for the slow subsystem. Pioneering work in this area was published by Menon et al. [14] in 1987. Reference [14] designed a flight-test trajectory control system using dynamic inversion. The output variables to be tracked were total velocity, angle of attack, sideslip, and altitude. Once the desired angular rates were calculated, the dynamic inversion was applied to the fast subsystem to compute the aerodynamic control surface deflections. This work was extended to overactuated systems by Snell et al. [15]. More recently, the same concept has been employed to design longitudinal windshear flight-control laws [16] and for control of generic reentry vehicles [17].

Although all of the systems studied fall under the category of Eqs. (1), different design methodologies have been developed for varied physical systems, and several different control techniques have been employed. The control laws developed for a general form of physical systems assume the existence of a unique solution of the transcendental equation. For general dynamical system models, the existence of isolated roots for the fast states is not guaranteed. Although aircraft literature addresses this problem by employing

assumptions about the plant model, there is no general methodology in the literature to date to design tracking control structures for singularly perturbed systems that are nonlinear, both in the slow and the fast states. The open-loop study of these systems has been the focus of the geometric singular perturbation theory [18]. This theory has been employed in the past for transforming dynamical systems into singular perturbation form [Eq. (1)] [19,20] and to develop reduced-order models [21]. Work by Sharkey and O'Reilly [22] used this approach to design stabilizing control laws for a special of class of singularly perturbed systems wherein the control appears only in the fast dynamics. The global nature of the preceding stabilization results was proved by Chen [23] later on in 1998.

In this paper, the use of geometric methods is extended to a general class of time-varying singularly perturbed systems that are nonlinear in both the slow and the fast states. The problem of control for this general class of singularly perturbed systems is addressed for the first time in a systematic manner. The paper makes two major contributions. First, this work is not restricted to systems that have a unique solution for the nonlinear algebraic set of equations of the slow subsystem. The presence of multiple roots is accounted for by proving that a center manifold exists for the slow subsystem. This allows for the incorporation of results from the center manifold theory that are helpful in obtaining approximate roots of the transcendental equations. Tracking control laws are designed for both the slow and the fast subsystems to track the desired reference and computed approximation, respectively, using a composite control methodology. Second, the composite control law is not a function of the scalar perturbation parameter, nor does it require knowledge of it. This is an important consideration for systems such as aircraft, where quantifying this parameter can be difficult. The proposed control scheme is able to guarantee asymptotic stabilization of states for a general class of nonlinear regulation problems and uniform bounded stability for the trajectory-following problem. Using Lyapunov theory, a conservative upper bound ϵ^* is derived for the singular perturbation parameter for which these results hold. From the stability analysis, it is shown that this approach applies to all classes of singularly perturbed systems, with tracking properties of standard singular perturbation models being a special case. The approach and methodology is demonstrated with simulation examples for a nonlinear generic two-degree-of-freedom kinetic model and a nonlinear, coupled six-degree-of-freedom F/A-18A Hornet aircraft.

The paper is organized as follows. Section II describes the class of systems considered and formulates the control problem. Section III presents the necessary concepts of geometric singular perturbation theory and motivates this work. Section IV makes an important observation about the existence of a center manifold for the singularly perturbed system and details the procedure to compute this manifold. Section V develops the reduced-order models and formulates the tracking control laws. The proof of stability and main results are also presented in this section. Numerical simulations are presented in Sec. VI, and conclusions are discussed in Sec. VII.

II. Problem Formulation

The dynamic system considered is the nonlinear affine in the control singularly perturbed system, mathematically expressed as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) + \mathbf{g}(\mathbf{x}, \mathbf{z})\mathbf{u} \tag{4}$$

$$\epsilon\dot{\mathbf{z}} = l(\mathbf{x}, \mathbf{z}) + \mathbf{k}(\mathbf{x}, \mathbf{z})\mathbf{u} \tag{5}$$

where $\mathbf{x} \in \mathbb{R}^m$ is the set of slow variables of the system, $\mathbf{z} \in \mathbb{R}^n$ is the vector of the fast variables, and $\mathbf{u} \in \mathbb{R}^p$ is the set of the control variable. The singular perturbation parameter satisfies $0 < \epsilon \ll 1$ and $\epsilon \in \mathbb{R}^+$. The vector fields $\mathbf{f}(.), \mathbf{g}(.), l(.)$, and $\mathbf{k}(.)$ are such that the closed-loop system is twice continuously differentiable with respect to their arguments. The control objective is to control the slow state to asymptotically track a specified twice continuously differentiable time-varying bounded trajectory, or $\mathbf{x}(t) \rightarrow \mathbf{x}_r(t)$ as $t \rightarrow \infty$.

Remark 1: The functions $\mathbf{g}(\mathbf{x}, \mathbf{z})$ and $\mathbf{k}(\mathbf{x}, \mathbf{z})$ represent the control-influence terms, while all other terms such as inertial coupling and gravitational forces are all contained in $\mathbf{f}(\mathbf{x}, \mathbf{z})$ and $l(\mathbf{x}, \mathbf{z})$.

Remark 2: For a rigid body, $\mathbf{x}$ are the translational velocities while $\mathbf{z}$ represents the angular velocities. The rotational dynamics for a rigid body contain the nonlinear inertial coupling terms. The function $l(\mathbf{x}, \mathbf{z})$ captures this nonlinearity in the fast states.

III. Background: Geometric Singular Perturbation Theory

Singular perturbation theory is a tool used to obtain the reduced-order approximations of the full-order equations of motion, which are difficult to analyze. The theory is valid so long as the parameter ϵ remains sufficiently small and the time-scale behavior is preserved. The method of matched asymptotic expansions [24] and its variation, the method of composite expansions [24], have been the foremost methods employed to develop these reduced-order models. The alternative geometric approach describes the motion of the full-order system using the concept of invariant manifolds. Both approaches produce the exact same reduced-order models but with different assumptions about the system. Asymptotic methods assume that the dynamical system possesses isolated roots, while the geometric approach is more general and takes into consideration multiple nonisolated roots of nonlinear systems.

To introduce the necessary concepts of geometric singular perturbation theory for an open-loop dynamical system, consider the nonlinear autonomous system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) \tag{6}$$

$$\epsilon\dot{\mathbf{z}} = l(\mathbf{x}, \mathbf{z}) \tag{7}$$

Note that the following results also apply to nonautonomous systems. Equations (6) and (7) can be rewritten in the fast time scale $\tau = (t - t_0)/\epsilon$ as

$$\mathbf{x}' = \epsilon\mathbf{f}(\mathbf{x}, \mathbf{z}) \tag{8}$$

$$\mathbf{z}' = l(\mathbf{x}, \mathbf{z}) \tag{9}$$

The independent variables t and τ are referred to as the slow and the fast time scales, respectively, and Eqs. (6–9) (referred to as the slow and the fast systems, respectively) are equivalent whenever $\epsilon \neq 0$. First, the system is studied for $\epsilon = 0$. The fast system reduces to n dimensions with variables $\mathbf{x}$ as constant parameters, producing the reduced fast subsystem,

$$\mathbf{x}' = \mathbf{0} \tag{10}$$

$$\mathbf{z}' = l(\mathbf{x}, \mathbf{z}) \tag{11}$$

On the other hand, the order of the slow system reduces to m dimensions and results in a set of differential-algebraic equations, producing the reduced slow subsystem,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) \tag{12}$$

$$\mathbf{0} = l(\mathbf{x}, \mathbf{z}) \tag{13}$$

The reduced slow system appears to be a locally flattened vector space of the complete slow system. Thus, the set of points $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}^m \times \mathbb{R}^n$ is expected to have a $\mathcal{C}^r$ smooth manifold $\mathcal{M}_0$ of dimension m inside the zero set of function $l(.)$, provided the functions $\mathbf{f}(.)$ and $l(.)$ are assumed to be $\mathcal{C}^r$.

Assumption 1: The functions $\mathbf{f}(\mathbf{x}, \mathbf{z})$ and $\boldsymbol{l}(\mathbf{x}, \mathbf{z})$ are sufficiently smooth so that $\mathcal{C}^r$ with $r \geq 1$.

The requirement to be continuous and at least once differentiable assures smoothness of the manifold $\mathcal{M}_0$. The flow on this manifold evolves as

$$\dot{\mathbf{x}} = \mathbf{f}[\mathbf{x}, \mathbf{h}_0(\mathbf{x})] \qquad (14)$$

where $\mathbf{h}_0(\mathbf{x})$ is the solution of the algebraic part [Eq. (13)] that defines the manifold,

$$\mathcal{M}_0: \mathbf{z} = \mathbf{h}_0(\mathbf{x}); \qquad \mathbf{x} \in \mathbb{R}^m, \qquad \mathbf{z} \in \mathbb{R}^n \qquad (15)$$

When viewed from the perspective of the reduced fast subsystem, the manifold $\mathcal{M}_0$ is the set of fixed points $[\mathbf{x}, \mathbf{h}_0(\mathbf{x})]$; therefore, $\mathcal{M}_0$ is trivially invariant. If every fixed point $[\mathbf{x}, \mathbf{h}_0(\mathbf{x})]$ of the reduced fast subsystem is assumed to be hyperbolic, then starting from arbitrary initial conditions, the flow will settle down exponentially fast onto the manifold, after which the flow evolves according to Eq. (14). Equivalently, the flow normal to the manifold is faster than that tangential to it. Such a manifold is said to be normally hyperbolic. Furthermore, a normally hyperbolic invariant manifold has local, $\mathcal{C}^r$ smooth stable, and unstable manifolds: $\mathcal{W}_{\text{loc}}^S(\mathcal{M}_0)$ and $\mathcal{W}_{\text{loc}}^U(\mathcal{M}_0)$. These manifolds are unions over all $(\mathbf{x})$ in $\mathcal{M}_0$ of the local stable and unstable manifolds of the reduced fast subsystem's hyperbolic fixed points $[\mathbf{x}, \mathbf{h}_0(\mathbf{x})]$.

To show these concepts, consider the following example. Let

$$\dot{x}_1 = -x_1 \qquad \dot{x}_2 = -x_2 \qquad \epsilon \dot{z} = -z \qquad (16)$$

so that the reduced slow subsystem is

$$\dot{x}_1 = -x_1 \qquad \dot{x}_2 = -x_2 \qquad -z = 0 \qquad (17)$$

and the reduced fast subsystem is

$$x_1 = 0 \qquad x_2 = 0 \qquad z' = -z \qquad (18)$$

The solution of the algebraic equation (17) is $z = 0$, which is also the fixed point of Eq. (18). The invariant manifold is given by $\mathcal{M}_0: z = 0$, which is the complete x_1-x_2 plane. The origin is the stable hyperbolic equilibrium of the reduced slow subsystem, so any trajectory starting on the manifold approaches the origin in forward time, as seen in Fig. 1. Studying the reduced fast subsystem suggests that, for any point with nonzero initial condition $z(0)$, the flow approaches normal to the manifold. Intuitively, one may conclude that, for initial conditions not on the manifold, the reduced fast subsystem describes the transition to the manifold, after which the system evolves according to the reduced slow subsystem (seen in Fig. 2). Furthermore, since all points (x_1, x_2, z) approach the manifold at an exponential rate forward in time, the complete space is the stable manifold $\mathcal{W}^S(\mathcal{M}_0)$.

For the full-order system, similar inferences can be made. The presence of ϵ in Eq. (7) indicates that the fast variables grow relatively faster than the other states of the system. If their open-loop system is stabilizing, these states quickly settle down to their equilibrium. The other variables continue to evolve in time with the fast variables fixed by an equilibrium hypersurface. Mathematically, $\exists \ t^*: t^* > t_0$, after which the solutions $\mathbf{x}(t, \epsilon)$ and $\mathbf{z}(t, \epsilon)$ lie on a distinct m dimensional-invariant manifold $\mathcal{M}_\epsilon$:

$$\mathcal{M}_\epsilon: \mathbf{z} = \mathbf{h}(\mathbf{x}, \epsilon); \qquad \mathbf{x} \in \mathbb{R}^m, \qquad \mathbf{z} \in \mathbb{R}^n \qquad (19)$$

For the system of Eqs. (16), the invariant manifold continues to be the x_1-x_2 plane. In addition, the family of lines parallel to the z axes still describe the flow normal to the manifold. Consider Fig. 3 to study this behavior. To generate this figure, ϵ was chosen to be 0.05. For a fixed initial condition, the flow evolves in two parts: one component along the manifold $\mathcal{M}_\epsilon$, which is governed by the reduced slow subsystem, and the other component in the normal direction, for which the flow is governed by the reduced fast subsystem. Points that are already on the manifold are seen to evolve similar to the flow sketched in Fig. 1. Thus, the reduced-order models
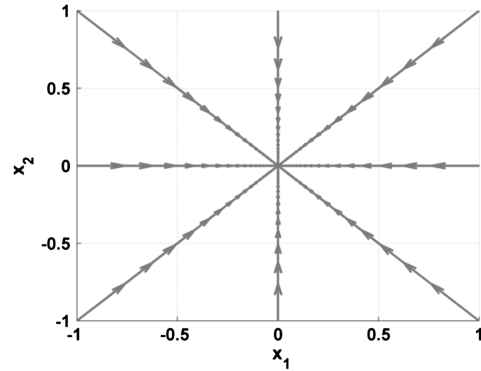
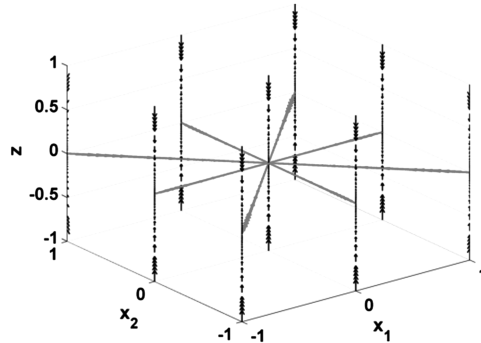

Fig. 1   Reduced slow subsystem.

Fig. 2   The reduced slow and the fast subsystems.

provide good insight into the behavior of the full-order system. It is apparent that if the reduced fast subsystem were unstable, then an initial condition not on the manifold would move farther away in time. For the example considered, the manifolds $\mathcal{M}_0$ and $\mathcal{M}_\epsilon$ were obtained to be identically equal, but this is not generally the case with nonlinear systems.

The geometric constructs discussed previously are formal statements of Fenichel's persistence theory [18]. First, the following assumptions about the slow system are made:

Assumption 2: There exists a set $\mathcal{M}_0$ that is contained in $\{(\mathbf{x}, \mathbf{z}): \boldsymbol{l}(\mathbf{x}, \mathbf{z}) = \mathbf{0}\}$, such that $\mathcal{M}_0$ is a compact boundaryless manifold.

Assumption 3: $\mathcal{M}_0$ is normally hyperbolic relative to the reduced fast subsystem and, in particular, it is required that for all points $\mathbf{z} \in \mathcal{M}_0$, there are k (respectively, l) eigenvalues of $D_\mathbf{z} \boldsymbol{l}(0, \mathbf{z})$ with positive (respectively, negative) real parts that are bounded away from zero, where $k + l = n$.

The following theorem from Fenichel [18] is for compact boundaryless manifolds. Let the slow system satisfy Assumptions 1, 2, and 3. If $\epsilon > 0$ is sufficiently small, then there exists a manifold $\mathcal{M}_\epsilon$ that is $\mathcal{C}^{r-1}$ smooth locally invariant under the fast system and $\mathcal{C}^{r-1}$ $\mathcal{O}(\epsilon)$ close to $\mathcal{M}_0$. In addition, there exist perturbed local stable and
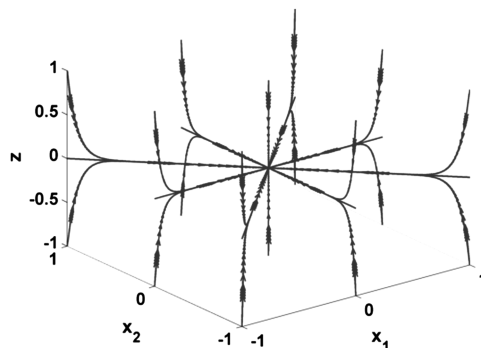

Fig. 3   Flow of system of equations (16) when $\epsilon \neq 0$.

unstable manifolds of $\mathcal{M}_\epsilon$, and they are $\mathcal{C}^r$ $\mathcal{O}(\epsilon)$ close, for all $r < \infty$, to their unperturbed counterparts.

IV.   Center Manifold and Computation

Fenichel's theorem [18] is a powerful tool to study the behavior of stiff dynamical systems. It asserts the presence of an invariant manifold $\mathcal{M}_\epsilon$ that is $\mathcal{O}(\epsilon)$ close to $\mathcal{M}_0$, but it does not provide the procedure to compute the manifold. Since $\mathcal{M}_\epsilon$ is invariant for some $t \geq t^*$, the solutions follow the curve specified in Eq. (19). Differentiating this expression with respect to t,

$$\dot{\mathbf{z}} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \dot{\mathbf{x}} \qquad (20)$$

and multiplying Eq. (20) with ϵ and substituting for $\dot{\mathbf{x}}$ and $\dot{\mathbf{z}}$ from Eqs. (6) and (7) results in

$$\epsilon \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \mathbf{f}[\mathbf{x}, \mathbf{h}(\mathbf{x}, \epsilon)] = l[\mathbf{x}, \mathbf{h}(\mathbf{x}, \epsilon)] \qquad (21)$$

Equation (21) is called the manifold condition. Note that substituting $\epsilon = 0$ in the manifold condition returns Eq. (13), which is satisfied by the manifold $\mathcal{M}_0$. To employ Fenichel's results [18], the manifold condition needs to be solved. Exact computation is impossible, since solving this condition is equivalent to solving the complete nonlinear system. One approximate approach is to substitute a perturbation expansion for $\mathbf{h}(\mathbf{x}, \epsilon) = \mathbf{h}_0(\mathbf{x}) + \epsilon \mathbf{h}_1(\mathbf{x}) + \mathcal{O}(\epsilon^2)$ into Eq. (21) and then solve order by order for $\mathbf{h}(\mathbf{x}, \epsilon)$. If the domain of interest is known, then the implicit function theorem may be employed. It is usually the inverse problem that is encountered, which is to find $\mathbf{h}(\mathbf{x}, \epsilon)$ as a smooth function of its arguments. In this paper, the approach proposed in [25] is used, and the following discusses its computation procedure.

The computation procedure proposed in [25] has been laid out for dynamical systems with center manifolds. For completeness, the first step is to check whether the manifold $\mathcal{M}_\epsilon$ is the center manifold of the singularly perturbed system. To study this behavior, rewrite the fast system using the technique called suspension [26] as

$$\mathbf{x}' = \epsilon \mathbf{f}(\mathbf{x}, \mathbf{z}) \qquad (22)$$

$$\epsilon' = 0 \qquad (23)$$

$$\mathbf{z}' = l(\mathbf{x}, \mathbf{z}) \qquad (24)$$

Assume that the origin is the fixed point of the preceding system that is $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $l(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Then, the perturbed system obtained by linearizing these equations about the origin [$\epsilon = 0$, $\mathbf{x} = \mathbf{0}$, $\mathbf{h}(\mathbf{0}, 0) = \mathbf{0}$] is written in compact form as

$$\Delta \mathbf{w}' = F\mathbf{w} + F_1 \mathbf{z} \qquad \Delta \mathbf{z}' = L\mathbf{z} + L_1 \mathbf{w} \qquad (25)$$

where $\mathbf{w} = [\mathbf{x}, \epsilon]^T$, $\Delta \mathbf{w}$, and $\Delta \mathbf{z}$ denote the perturbation quantities, and F, F_1, L, and L_1 are constant matrices of appropriate size. If all eigenvalues of F have zero real parts while all eigenvalues of L have negative real parts, then the manifold $\mathcal{M}_\epsilon$ is precisely the center manifold, and it spans the generalized eigenvectors associated with eigenvalues with zero real parts. This manifold is defined for all small values of the slow state $\mathbf{x}$ and the perturbation parameter ϵ. The requirement on eigenvalues of F supports the existence of time scales in the system, for if the eigenvalues were nonzero, then all states would be fast variables, and the system is not singularly perturbed. This suggests that the eigenvalue restriction on F is always satisfied by systems with the multiple time-scale property. The other requirement of negative eigenvalues of L is to ensure that the trajectories not on the manifold approach it in forward time.

From the preceding analysis, $\mathbf{h}(\mathbf{x}, \epsilon)$ is known to be the center manifold. If the origin is the fixed point of the linearized system, then the theorem from [25] asserts that one can approximate $\mathbf{h}(\mathbf{x}, \epsilon)$ to any

degree of accuracy. For functions $\phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, which are $\mathcal{C}^{r-1}$ (r defined as in Assumption 1) in the neighborhood of the origin, define

$$(M\phi)(\mathbf{x}, \epsilon) = \epsilon \frac{\partial \phi}{\partial \mathbf{x}} \mathbf{f}[\mathbf{x}, \phi(\mathbf{x}, \epsilon)] - l[\mathbf{x}, \phi(\mathbf{x}, \epsilon)] \qquad (26)$$

Note that, by Eq. (21), $(M\mathbf{h})(\mathbf{x}, \epsilon) = \mathbf{0}$.

The following is the theorem from [25]. Let $\phi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ satisfy $\phi(\mathbf{0}, 0) = \mathbf{0}$ and $|(M\phi)(\mathbf{x}, \epsilon)| = \mathcal{O}[C(\mathbf{x}, \epsilon)]$ for $|\mathbf{x}| \to 0$ and $\epsilon \to 0$, where $C(.)$ is a polynomial of degree greater than one. Then,

$$|\mathbf{h}(\mathbf{x}, \epsilon) - \phi(\mathbf{x}, \epsilon)| = \mathcal{O}[C(\mathbf{x}, \epsilon)]$$

This theorem implies that an approximate function $\phi(\mathbf{x}, \epsilon)$ can be determined for small values of $\mathbf{x}$ and ϵ. The condition $\phi(\mathbf{0}, 0) = \mathbf{0}$ is to ensure that the origin remains the fixed point. To demonstrate the procedure, consider the example from [25]:

$$\dot{x} = xz + ax^3 + bz^2 x \qquad \epsilon \dot{z} = -z + cx^2 + dx^2 z \qquad (27)$$

Linearizing this system about the origin,

$$\Delta x' = 0 \qquad \Delta \epsilon' = 0 \qquad \Delta z' = -1 \qquad (28)$$

It is seen that the system possesses a center manifold $z = h(x, \epsilon)$. To approximate h, define

$$(M\phi)(x, \epsilon) = \epsilon \frac{\partial \phi}{\partial x}[x\phi(x, \epsilon) + ax^3 + b\phi^2(x, \epsilon)x] + \phi(x, \epsilon) - cx^2$$
$$- dx^2 \phi(x, \epsilon) \qquad (29)$$

Hence, if $\phi(x, \epsilon) = cx^2$, then $(M\phi)(x, \epsilon) = \mathcal{O}(|x^4| + |\epsilon x^4|)$, and from the preceding theorem, $h(x, \epsilon) = cx^2 + \mathcal{O}(|x^4| + |\epsilon x^4|)$. Since the fast subsystem is stabilizing, geometric singular perturbation theory says that stability of the complete system can be analyzed by studying the flow on the manifold [Eq. (14)]:

$$\dot{x} = (a + c)x^3 + bcx^5 + \mathcal{O}(|x^5| + |\epsilon x^5|) \qquad (30)$$

V.   Control Formulation and Stability Analysis

The central idea is the following. If the reduced fast subsystem is stabilizing about the manifold $\mathcal{M}_0$, the complete system dynamics remain $\mathcal{O}(\epsilon)$ close to the reduced slow subsystem. This fact is employed to develop a stable closed-loop system. It is proposed that two separate stabilizing controllers be designed for each of the subsystems and their composite be fed to the complete system. It is shown that, in fact, this composite control uniformly stabilizes the complete system. This approach has been shown in the literature to guarantee asymptotic stability for singularly perturbed systems with unique manifolds $\mathcal{M}_0$ [10]. In the following subsections, control laws for a general class of nonlinear singularly perturbed systems are formulated, and closed-loop system stability is analyzed.

A.   Control Law Development

The objective is to augment the two time-scale system with state feedback controllers such that the system follows a specified continuous twice differentiable bounded trajectory $\mathbf{x}_r(t)$. The first step is to transform the system [Eqs. (4) and (5)] into a non-autonomous stabilization problem. Define the error signal as $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_r(t)$. Then,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z}) + \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z})\mathbf{u} - \dot{\mathbf{x}}_r \qquad (31)$$

$$\epsilon \dot{\mathbf{z}} = l(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z}) + \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z})\mathbf{u} \qquad (32)$$

The objective is to seek the control vector of the form $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f$, where

$$\mathbf{u}_s = \Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) \qquad (33)$$

and

$$\mathbf{u}_f = \Gamma_f(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r) \qquad (34)$$

Substituting the controls into Eqs. (31) and (32) produces

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z}) + \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z})[\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$+ \Gamma_f(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r)] - \dot{\mathbf{x}}_r \qquad (35)$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{l}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z}) + \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{z})[\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \Gamma_f(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{x}_r, \dot{\mathbf{x}}_r)] \qquad (36)$$

Assume that the right-hand side of Eqs. (35) and (36) is $\mathcal{C}^2$; that is, the vector fields satisfy Assumption 1 with $r = 2$. From Fenichel's theorem [18], it can be concluded that there exists a manifold

$$\mathcal{M}_\epsilon : \mathbf{z} = \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r) \qquad (37)$$

that satisfies the manifold condition,

$$\epsilon\frac{\partial\mathbf{h}}{\partial t} + \epsilon\frac{\partial\mathbf{h}}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} = \mathbf{l}[\tilde{\mathbf{x}} + \mathbf{x}_r, \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)] + \mathbf{k}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{k}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_f[\tilde{\mathbf{x}}, \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r), \mathbf{x}_r, \dot{\mathbf{x}}_r] \qquad (38)$$

Note that the manifold is time dependent, since the system under consideration is nonautonomous due to the time varying $\mathbf{x}_r(t)$. Define the error between the fast states and the manifold $\mathcal{M}_\epsilon$ as $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)$.

The transformed system with the origin as the equilibrium is expressed as

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)] + \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_f[\tilde{\mathbf{x}}, \tilde{\mathbf{z}} + \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r), \mathbf{x}_r, \dot{\mathbf{x}}_r] - \dot{\mathbf{x}}_r \qquad (39)$$

$$\epsilon\dot{\tilde{\mathbf{z}}} = \mathbf{l}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)] + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\Gamma_f[\tilde{\mathbf{x}}, \tilde{\mathbf{z}} + \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r), \mathbf{x}_r, \dot{\mathbf{x}}_r]$$
$$- \epsilon\frac{\partial\mathbf{h}}{\partial t} - \epsilon\frac{\partial\mathbf{h}}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} \qquad (40)$$

Note that the error $\tilde{\mathbf{z}} = \mathbf{0}$ when the manifold condition is satisfied. It is known that the exact manifold $\mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)$ is impossible to compute. Let $\phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$ be an approximate manifold obtained using the procedure presented in Sec. IV. The approximate manifold is chosen to contain terms independent of ϵ, similar to the example considered at the end of Sec. IV. Define

$$(M\phi)(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s, \Gamma_f) = \epsilon\frac{\partial\phi}{\partial t} + \epsilon\frac{\partial\phi}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}}$$
$$- \mathbf{l}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]$$
$$- \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$- \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s))\Gamma_f(\tilde{\mathbf{x}}, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s), \mathbf{x}_r, \dot{\mathbf{x}}_r) \qquad (41)$$

and let $(M\phi)(t, \tilde{\mathbf{x}}, \epsilon) = \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]$, which depends on the choice of controls Γ_s and Γ_f. Furthermore, the following is assumed:

Assumption A: The choice of controls Γ_s and Γ_f leads to $\mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon = 0, \mathbf{x}_r, \dot{\mathbf{x}}_r)] = \mathbf{0}$.

With the preceding choice of $\phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$, the exact manifold is given as

$$\mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r) = \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]$$

Substituting the approximate expression for the manifold into Eqs. (39) and (40),

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} + \mathbf{g}\{\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$+ \mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \dot{\mathbf{x}}_r \qquad (42)$$

$$\epsilon\dot{\tilde{\mathbf{z}}} = \mathbf{l}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} + \mathbf{k}\{\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r]\})\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$+ \mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$- \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial t} - \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} \qquad (43)$$

Note that Γ_f is a function of Γ_s due to the choice of $\phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$. The reduced slow and fast subsystems for the system of Eqs. (42) and (43) are obtained by substituting $\epsilon = 0$, resulting in the reduced slow subsystem,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \dot{\mathbf{x}}_r \qquad (44)$$

$$\mathbf{0} = \mathbf{l}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \qquad (45)$$

and the reduced fast subsystem,

$$\tilde{\mathbf{x}}' = \mathbf{0} \qquad (46)$$

$$\tilde{\mathbf{z}}' = \mathbf{l}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \qquad (47)$$

In general, the composite control approach first computes the control Γ_s required to maintain reduced slow subsystem stability by assuming that the fast states lie upon the manifold and $\Gamma_f = \mathbf{0}$. In the next step, the control Γ_f is designed to satisfy two conditions: guarantee uniform convergence of the fast states onto the manifold and remain inactive when the fast state remains on the manifold. The second condition is implemented to avoid affecting the conclusions drawn about the reduced slow subsystem stability. In the proposed control scheme, the second condition is avoided by designing Γ_f ahead of Γ_s. Thus, design $\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$ as a function of Γ_s, such that Eq. (47) is transformed into the closed-loop reduced fast subsystem,

$$\tilde{\mathbf{z}}' = -\mathbf{L}_\mathbf{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{K}_\mathbf{f}(\tilde{\mathbf{z}}) \qquad (48)$$

such that $-\mathbf{L}_\mathbf{f}(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{K}_\mathbf{f}(\mathbf{0}) = \mathbf{0}$. With this choice of Γ_f and assumptions about vector fields $\mathbf{L}_\mathbf{f}$ and $\mathbf{K}_\mathbf{f}$, $\tilde{\mathbf{z}} = \mathbf{0}$ becomes the

unique root of Eq. (45). Therefore, the reduced slow subsystem reduces to

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]$$
$$+ \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$+ \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \dot{\mathbf{x}}_r \quad (49)$$

The only unknown in Eq. (49) is Γ_s; therefore, it may be designed to transform the reduced slow subsystem into the closed-loop reduced slow subsystem,

$$\dot{\tilde{\mathbf{x}}} = -\mathbf{F_s}(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{G_s}(\tilde{\mathbf{x}}) \quad (50)$$

and exact forms of $\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$, $\phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$, and correspondingly $C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)$, can be determined through Eqs. (48) and (41), respectively.

Remark 3: In the reduced subsystems obtained, $\tilde{\mathbf{z}} = \mathbf{z} - \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$ by Assumption A. Thus, at the implementation level, the control Γ_f is a function of known quantities.

The complete closed-loop system is obtained by rewriting Eqs. (42) and (43) as

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \mathbf{g}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{g}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \dot{\mathbf{x}}_r + \mathbf{f}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] - \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}_r, \phi[\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]$$
$$+ \{\mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] - \mathbf{g}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\}\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \mathbf{g}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathbf{f}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + (\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + (\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \quad (51)$$

$$\epsilon\dot{\tilde{\mathbf{z}}} = l[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + l\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - l[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + (\mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + (\mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$- \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial t} - \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} \quad (52)$$

Using the closed-loop reduced subsystems of Eqs. (48) and (50), Eqs. (51) and (52) become the closed-loop complete system,

$$\dot{\tilde{\mathbf{x}}} = -\mathbf{F_s}(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{G_s}(\tilde{\mathbf{x}}) + \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]$$
$$- \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + \{\mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\}\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$$
$$+ \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \mathbf{g}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathbf{f}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] + (\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + (\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$
$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \quad (53)$$

$$\epsilon\dot{\tilde{\mathbf{z}}} = -\mathbf{L_f}(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{K_f}(\tilde{\mathbf{z}}) + l\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$+ \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - l[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]$$
$$+ (\mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}$$
$$- \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + (\mathbf{k}\{\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{k}[\tilde{\mathbf{x}}$$
$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)$$
$$- \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial t} - \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial\tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} \quad (54)$$

Remark 4: If $\phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$ is the unique manifold for the complete system, then the terms of $\mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]$ are identically zero, and the closed-loop complete system in Eqs. (53) and (54) takes the form as in [27,28], which has been proven to be closed-loop stable.

B. Stability Analysis

1. Tracking Problem

The following theorem summarizes the main result of the paper.

Theorem 1: Suppose the controls $\mathbf{u}_s$ and $\mathbf{u}_f$ are designed according to Eqs. (48) and (50) and Assumptions A–I hold. Then for all initial conditions, $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \in D_x \times D_z$, the composite control $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f$ uniformly stabilizes the nonlinear singularly perturbed system in Eqs. (4) and (5) for all $\epsilon < \epsilon^*$, where ϵ^* is given by the inequality equation (68), and the error signals $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{z}}(t)$ are uniformly bounded by Eqs. (69) and (70), respectively.

Proof: Closed-loop system stability is analyzed using the composite Lyapunov function approach [29]. It is required to prove that the closed-loop system behavior remains close to the closed-loop reduced slow subsystem. Suppose that there are Lyapunov functions $V(t, \tilde{\mathbf{x}}) = \frac{1}{2}\tilde{\mathbf{x}}^T\tilde{\mathbf{x}}$ and $W(t, \tilde{\mathbf{z}}) = \frac{1}{2}\tilde{\mathbf{z}}^T\tilde{\mathbf{z}}$ for the closed-loop reduced-order models (50) and (48), respectively, satisfying the following eight assumptions:

B) $V(t, \tilde{\mathbf{x}})$ is positive definite and decrescent; that is,

$$c_1\|\tilde{\mathbf{x}}\|^2 \leq V(t, \tilde{\mathbf{x}}) \leq c_2\|\tilde{\mathbf{x}}\|^2, \qquad \tilde{\mathbf{x}} \in D_x \subset \mathbb{R}^m \quad (55)$$

C)

$$\frac{\partial V}{\partial\tilde{\mathbf{x}}}[-\mathbf{F_s}(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{G_s}(\tilde{\mathbf{x}})] \leq -\alpha_1\|\tilde{\mathbf{x}}\|^2 - b_1\|\tilde{\mathbf{x}}\|, \qquad \alpha_1 > 0,$$
$$b_1 \geq 0 \quad (56)$$

D) There exists a constant $\beta_1 > 0$, such that

$$\frac{\partial V}{\partial \tilde{\mathbf{x}}}\{\mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] - \mathbf{f}[\tilde{\mathbf{x}} + \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\}$$

$$+ \frac{\partial V}{\partial \tilde{\mathbf{x}}}\{\mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)] - \mathbf{g}[\tilde{\mathbf{x}}$$

$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\}\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \frac{\partial V}{\partial \tilde{\mathbf{x}}}\{\mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) - \mathbf{g}[\tilde{\mathbf{x}}$$

$$+ \mathbf{x}_r, \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]\Gamma_f(\tilde{\mathbf{x}}, \mathbf{0}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)\} \leq \beta_1 \|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| \quad (57)$$

E) There exist constants $\beta_2 > 0$, $\beta_3 > 0$, and $\beta_4 \geq 0$, such that

$$\frac{\partial V}{\partial \tilde{\mathbf{x}}}(\mathbf{f}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{f}[\tilde{\mathbf{x}}$$

$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]) + \frac{\partial V}{\partial \tilde{\mathbf{x}}}(\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \frac{\partial V}{\partial \tilde{\mathbf{x}}}(\mathbf{g}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{g}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \leq \epsilon\beta_2\|\tilde{\mathbf{x}}\|^2$$

$$+ \epsilon\beta_3\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| + \epsilon\beta_4\|\tilde{\mathbf{x}}\| \quad (58)$$

F) $W(t, \tilde{\mathbf{z}})$ is positive definite and decrescent scalar function satisfying,

$$c_3\|\tilde{\mathbf{z}}\|^2 \leq W(t, \tilde{\mathbf{z}}) \leq c_4\|\tilde{\mathbf{z}}\|^2, \qquad \tilde{\mathbf{z}} \in D_z \subset \mathbb{R}^n \quad (59)$$

G)

$$\frac{\partial W}{\partial \tilde{\mathbf{z}}}[-\mathbf{L_f}(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \mathbf{K_f}(\tilde{\mathbf{z}})] \leq -\alpha_2\|\tilde{\mathbf{z}}\|^2, \qquad \alpha_2 > 0 \quad (60)$$

H) There exist scalars $\beta_5 > 0$, $\beta_6 > 0$, and $\beta_7 \geq 0$, such that

$$\frac{\partial W}{\partial \tilde{\mathbf{z}}}(l\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - l[\tilde{\mathbf{x}}$$

$$+ \mathbf{x}_r, \tilde{\mathbf{z}} + \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)]) + \frac{\partial W}{\partial \tilde{\mathbf{z}}}(\mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{k}[\tilde{\mathbf{x}}, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_s(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r) + \frac{\partial W}{\partial \tilde{\mathbf{z}}}(\mathbf{k}\{\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\} - \mathbf{k}[\tilde{\mathbf{x}} + \mathbf{x}_r, \tilde{\mathbf{z}}$$

$$+ \phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s)])\Gamma_f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_s) \leq \epsilon\beta_5\|\tilde{\mathbf{z}}\|^2$$

$$+ \epsilon\beta_6\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| + \epsilon\beta_7\|\tilde{\mathbf{z}}\| \quad (61)$$

I) There exist constants $\beta_8 \geq 0$ and $\beta_9 > 0$, such that

$$-\frac{\partial W}{\partial \tilde{\mathbf{z}}}\left[\epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial t}\right.$$

$$\left.+ \epsilon\frac{\partial\{\phi + \mathcal{O}[C(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r)]\}}{\partial \tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}}\right] \leq \epsilon\beta_8\|\tilde{\mathbf{z}}\| + \epsilon\beta_9\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| \quad (62)$$

Remark 5: Assumptions B, C, F, and G are conditions for asymptotic stability of closed-loop reduced-order models. The constant b_1 in Assumption C depends upon the bounds of the specified trajectory $\mathbf{x}_r(t)$ and its derivative $\dot{\mathbf{x}}_r$. If the control Γ_s is designed to maintain asymptotic stability of the closed-loop slow subsystem, then $b_1 = 0$. Additionally, Assumptions D, E, H, and I are interconnection conditions obtained by assuming the vector fields are locally Lipschitz. The constants β_4, β_7, and β_8 appear due to the time-varying nature of the manifold and depend upon the bounds of $\mathbf{x}_r(t)$ and its derivative $\dot{\mathbf{x}}_r$. The constant β_8 also depends upon the

derivative $\ddot{\mathbf{x}}_r$, which is known to be bounded by the choice of the reference trajectory. Consider the Lyapunov function candidate,

$$v(t, \tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = V(t, \tilde{\mathbf{x}}) + W(\mathbf{t}, \tilde{\mathbf{z}}) \quad (63)$$

for the closed-loop system of Eqs. (53) and (54). From the properties of V and W, it follows that $v(t, \tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ is positive definite and decrescent. The derivative of v along the trajectories of Eqs. (53) and (54) is given by

$$\dot{v} = \frac{\partial V}{\partial \tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} + \frac{1}{\epsilon}\frac{\partial W}{\partial \tilde{\mathbf{z}}}\tilde{\mathbf{z}}' \quad (64)$$

Substituting Assumptions B–I into Eq. (64),

$$\dot{v} \leq -\alpha_1\|\tilde{\mathbf{x}}\|^2 - b_1\|\tilde{\mathbf{x}}\| + \beta_1\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| + \epsilon\beta_2\|\tilde{\mathbf{x}}\|^2 + \epsilon\beta_3\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\|$$

$$+ \epsilon\beta_4\|\tilde{\mathbf{x}}\|$$

$$- \frac{\alpha_2}{\epsilon}\|\tilde{\mathbf{z}}\|^2 - \beta_5\|\tilde{\mathbf{z}}\|^2 + \beta_6\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| + \beta_7\|\tilde{\mathbf{z}}\| + \beta_8\|\tilde{\mathbf{z}}\|$$

$$+ \beta_9\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\| \quad (65)$$

Collecting like terms

$$\dot{v} \leq -(\alpha_1 - \epsilon\beta_2)\|\tilde{\mathbf{x}}\|^2 - (b_1 - \epsilon\beta_4)\|\tilde{\mathbf{x}}\| + (\beta_1 + \epsilon\beta_3 + \beta_6$$

$$+ \beta_9)\|\tilde{\mathbf{x}}\|\|\tilde{\mathbf{z}}\|$$

$$- \left(\frac{\alpha_2}{\epsilon} - \beta_5\right)\|\tilde{\mathbf{z}}\|^2 - (-\beta_7 - \beta_8)\|\tilde{\mathbf{z}}\| \quad (66)$$

Rearrange Eq. (66) to get

$$\dot{v} \leq \begin{bmatrix}\|\tilde{\mathbf{x}}\| \\ \|\tilde{\mathbf{z}}\|\end{bmatrix}^T \begin{bmatrix} -(1-d)(\alpha_1 - \epsilon\beta_2) & \frac{1}{2}(\beta_1 + \epsilon\beta_3 + \beta_6 + \beta_9) \\ \frac{1}{2}(\beta_1 + \epsilon\beta_3 + \beta_6 + \beta_9) & -(1-d)\left(\frac{\alpha_2}{\epsilon} - \beta_5\right)\end{bmatrix}$$

$$\times \begin{bmatrix}\|\tilde{\mathbf{x}}\| \\ \|\tilde{\mathbf{z}}\|\end{bmatrix} - \|\tilde{\mathbf{x}}\|\{d(\alpha_1 - \epsilon\beta_2)\|\tilde{\mathbf{x}}\| - (\epsilon\beta_4 - b_1)\}$$

$$- \|\tilde{\mathbf{z}}\|\left\{d\left(\frac{\alpha_2}{\epsilon} - \beta_5\right)\|\tilde{\mathbf{z}}\| - (\beta_7 + \beta_8)\right\}, \qquad 0 < d < 1 \quad (67)$$

The matrix becomes negative definite when

$$(1 - d)^2(\alpha_1 - \epsilon\beta_2)\left(\frac{\alpha_2}{\epsilon} - \beta_5\right) < \frac{1}{4}(\beta_1 + \epsilon\beta_3 + \beta_6 + \beta_9)^2 \quad (68)$$

Thus, there exists an upper bound ϵ^* and upper bounds on the errors $\tilde{\mathbf{x}}_b$ and $\tilde{\mathbf{z}}_b$,

$$\tilde{\mathbf{x}}_b = \frac{\epsilon\beta_4 - b_1}{d(\alpha_1 - \epsilon\beta_2)} \quad (69)$$

$$\tilde{\mathbf{z}}_b = \frac{\beta_7 + \beta_8}{d(\frac{\alpha_2}{\epsilon} - \beta_5)} \quad (70)$$

for which

$$\dot{v} \leq 0 \quad (71)$$

From the Lyapunov theorem, it can then be concluded that the closed-loop signals $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{z}}$ are uniformly bounded for all initial conditions $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \in D_x \times D_z$. Consequently, the control vector $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f$ is bounded. Furthermore, since the trajectory $\mathbf{x}_r(t)$ is bounded, the manifold $\mathbf{h}(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)$ and the closed-loop signals $\mathbf{x}(t)$ and $\mathbf{z}(t)$ are bounded. □

2. Special Case: Regulation Problem

The following theorem summarizes the main result for the stabilization problem.

Theorem 2: Suppose the controls $\mathbf{u}_s$ and $\mathbf{u}_f$ are designed according to Eqs. (48–50), and Assumptions A–I hold with $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{z}} = \mathbf{z}$. Then, for all initial conditions $(\mathbf{x}, \mathbf{z}) \in D_x \times D_z$, the composite control $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f$ asymptotically stabilizes the nonlinear singularly perturbed systems in Eqs. (4) and (5) for all $\epsilon < \epsilon_s^*$, where ϵ_s^* is given by the inequality equation (68) with $d = 0$.

Proof: Note that, in this case, the manifold $\mathbf{h}(\mathbf{x}, \epsilon)$ is not time varying, and $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{z}} = \mathbf{z}$. Since this problem is autonomous, the decrescent conditions on the Lyapunov functions V and W can be relaxed. The constants β_4, β_7, and β_8 in Assumptions E, H, and I are all equal to zero, and the constant $b_1 = 0$, since $\mathbf{x}_r = \mathbf{0}$ and $\dot{\mathbf{x}}_r = \mathbf{0}$. With these modifications, Eq. (67) is modified as

$$\dot{v} \le \begin{bmatrix} \|\mathbf{x}\| \\ \|\mathbf{z}\| \end{bmatrix}^T \begin{bmatrix} -(\alpha_1 - \epsilon\beta_2) & \frac{1}{2}(\beta_1 + \epsilon\beta_3 + \beta_6 + \beta_9) \\ \frac{1}{2}(\beta_1 + \epsilon\beta_3 + \beta_6 + \beta_9) & -\left(\frac{\alpha_2}{\epsilon} - \beta_5\right) \end{bmatrix}$$
$$\times \begin{bmatrix} \|\mathbf{x}\| \\ \|\mathbf{z}\| \end{bmatrix} \tag{72}$$

Therefore, there exists an ϵ_s^* such that

$$\dot{v} < 0 \tag{73}$$

where ϵ_s^* satisfies the inequality equation (68) with $d = 0$. $\square$

Remark 6: Theorems 1 and 2 depend upon the approximation of the invariant manifold, leading to local results. If it were possible to obtain the expression of the exact manifold, these results would be valid globally.

Remark 7: Fenichel's theorem [18] implies that the behavior of the complete nonlinear system remains close to the reduced slow subsystem if the reduced fast subsystem is stable. Theorems 1 and 2 state the same result for the closed-loop singularly perturbed system.

VI. Numerical Examples

A. Purpose and Scope

The preceding theoretical developments are demonstrated with simulation. The first example is a generic planar nonlinear system. This planar example enables the study of the geometric constructs, which are generally difficult to visualize in higher-dimension problems. A step-by-step procedure of controller development is detailed for the system to track a desired slow kinetic state. A comparison between the manifold approximation and the attained actual fast state is made. The closed-loop results are studied for a sinusoidal time-varying trajectory and the regulator problem. The second example develops control laws for a nonlinear F/A-18A Hornet model. The objective of this example is to test the performance of the controller for a highly nonlinear, two time-scale system. It is required to perform a turning maneuver while maintaining zero sideslip and tracking a specified angle-of-attack profile.

B. Generic Two-Degree-of-Freedom Nonlinear Kinetic Model

The fast dynamics are modified to include an arbitrarily chosen quadratic nonlinearity in the fast state, and a pseudocontrol term with unit effectiveness is introduced. For this example, $x \in \mathbb{R}$ and $z \in \mathbb{R}$ represent the slow and the fast states, respectively. The control $u \in \mathbb{R}$ is developed to track a desired smooth trajectory $x_r(t)$:

$$\dot{x} = -x + (x + 0.5)z + u \tag{74}$$

$$\epsilon\dot{z} = x - (x + 1)z + z^2 + u \tag{75}$$

The value $\epsilon = 0.2$ is retained in the modified model [26].

1. Controller Design

Define the errors $\tilde{x} = x - x_r$ and $\tilde{z} = z - h(\tilde{x}, \epsilon, x_r, \dot{x}_r)$, and transform Eqs. (74) and (75) into error coordinates equivalent to Eqs. (39) and (40):

$$\dot{\tilde{x}} = -(\tilde{x} + x_r) + (\tilde{x} + x_r + 0.5)[\tilde{z} + h(\tilde{x}, \epsilon, x_r, \dot{x}_r)]$$
$$- \dot{x}_r + \Gamma_s + \Gamma_f \tag{76}$$

$$\epsilon\dot{\tilde{z}} = (\tilde{x} + x_r) - (\tilde{x} + x_r + 1)[\tilde{z} + h(\tilde{x}, \epsilon, x_r, \dot{x}_r)]$$
$$+ [\tilde{z} + h(\tilde{x}, \epsilon, x_r, \dot{x}_r)]^2 + \Gamma_s + \Gamma_f \tag{77}$$

Rearrange Eqs. (76) and (77), dropping arguments of h:

$$\dot{\tilde{x}} = -\tilde{x} + \tilde{x}h(.) + 0.5h(.) - x_r + x_r h(.) + (\tilde{x} + x_r + 0.5)\tilde{z} - \dot{x}_r$$
$$+ \Gamma_s + \Gamma_f \tag{78}$$

$$\epsilon\dot{\tilde{z}} = -(\tilde{x} + x_r + 1)\tilde{z} + \tilde{z}^2 + 2\tilde{z}h(.) + \tilde{x} + x_r - (\tilde{x} + x_r + 1)h(.)$$
$$+ h(.)^2 + \Gamma_s + \Gamma_f - \epsilon\frac{\partial h}{\partial t} - \epsilon\frac{\partial h}{\partial \tilde{x}}\dot{\tilde{x}} \tag{79}$$

Comparing with Eqs. (39) and (40),

$$\mathbf{f}(.) = -\tilde{x} + \tilde{x}h(.) + 0.5h(.) - x_r + x_r h(.) + (\tilde{x} + x_r + 0.5)\tilde{z}$$

$$\mathbf{g}(.) = 1$$

$$\mathbf{l}(.) = -(\tilde{x} + x_r + 1)\tilde{z} + \tilde{z}^2 + 2\tilde{z}h(.) + \tilde{x} + x_r - (\tilde{x} + x_r + 1)h(.)$$
$$+ h(.)^2$$

$$\mathbf{k}(.) = 1 \tag{80}$$

Let $\phi(\tilde{x}, x_r, \dot{x}_r, \Gamma_s)$ be the approximate manifold. Define the manifold condition:

$$(M\phi)(\tilde{x}, x_r, \dot{x}_r, \Gamma_s, \Gamma_f) = \epsilon\frac{\partial \phi}{\partial t} + \epsilon\frac{\partial \phi}{\partial \tilde{x}}\dot{\tilde{x}} - \tilde{x} - x_r + (\tilde{x} + x_r)\phi(.)$$
$$+ \phi(.) - \phi(.)^2 - \Gamma_s - \Gamma_f \tag{81}$$

Select

$$\phi(\tilde{x}, x_r, \dot{x}_r, \Gamma_s) = \tilde{x} + x_r + \Gamma_s \tag{82}$$

so that

$$(M\phi)(\tilde{x}, x_r, \dot{x}_r, \Gamma_s) = \epsilon\frac{\partial \phi}{\partial t} + \epsilon\frac{\partial \phi}{\partial \tilde{x}}\dot{\tilde{x}} + (\tilde{x} + x_r)(\tilde{x} + x_r + \Gamma_s)$$
$$- \phi(.)^2 - \Gamma_f \tag{83}$$

and

$$\mathcal{O}[C(\tilde{x}, \epsilon, x_r, \dot{x}_r)] = (M\phi)(\tilde{x}, x_r, \dot{x}_r, \Gamma_s)$$

To design the control Γ_f, develop the reduced fast subsystem equivalent to Eqs. (46) and (47):

$$\tilde{x}' = 0$$

$$\tilde{z}' = -(\tilde{x} + x_r + 1)\tilde{z} + \tilde{z}^2 + 2\tilde{z}\phi(.) + \tilde{x} + x_r - (\tilde{x} + x_r + 1)\phi(.)$$
$$+ \phi(.)^2 + \Gamma_s + \Gamma_f \tag{84}$$

Design

$$\Gamma_f = -A_f\tilde{z} - 2\tilde{z}\phi(.) + (\tilde{x} + x_r + 1)\phi(.) - \tilde{x} - x_r - \phi^2 - \Gamma_s \tag{85}$$

where A_f is a feedback gain. Then, the closed-loop reduced fast subsystem becomes

$$\tilde{z}' = -(\tilde{x} + x_r + 1 + A_f)\tilde{z} + \tilde{z}^2 \tag{86}$$

Comparing with Eq. (48),

$$\mathbf{L}_f(.) = (\tilde{x} + x_r + 1 + A_f)\tilde{z} \qquad \mathbf{K}_f(.) = \tilde{z}^2 \qquad (87)$$

The next step is to determine the control Γ_s. Develop the reduced-order slow subsystem equivalent to Eqs. (44) and (45):

$$\dot{\tilde{x}} = -\tilde{x} + \tilde{x}\phi(.) + 0.5\phi(.) - x_r + x_r\phi(.) + (\tilde{x} + x_r + 0.5)\tilde{z} - \dot{x}_r$$
$$+ \Gamma_s + \Gamma_f \qquad (88)$$

$$0 = -(\tilde{x} + x_r + 1)\tilde{z} + \tilde{z}^2 + 2\tilde{z}\phi(.) + \tilde{x} + x_r - (\tilde{x} + x_r + 1)\phi(.)$$
$$+ \phi(.)^2 + \Gamma_s + \Gamma_f \qquad (89)$$

Substituting for Γ_f from Eq. (85) in Eqs. (88) and (89),

$$\dot{\tilde{x}} = -2\tilde{x} + \tilde{x}\phi(.) + 0.5\phi(.) - 2x_r + x_r\phi(.) + (\tilde{x} + x_r + 0.5)\tilde{z}$$
$$- \dot{x}_r - \phi(.)^2 - 2\tilde{z}\phi(.) + (\tilde{x} + x_r + 1)\phi(.) - A_f\tilde{z} \qquad (90)$$

$$0 = -(\tilde{x} + x_r + 1 + A_f)\tilde{z} + \tilde{z}^2 \qquad (91)$$

Since $\tilde{z} = 0$ is the root of the algebraic solution, the reduced slow subsystem is obtained as

$$\dot{\tilde{x}} = -2\tilde{x} + \tilde{x}\phi(.) + 0.5\phi(.) - 2x_r + x_r\phi(.) - \dot{x}_r - \phi(.)^2$$
$$+ (\tilde{x} + x_r + 1)\phi(.) \qquad (92)$$

Substituting the expression for $\phi(.)$ from Eq. (82) in Eq. (92),

$$\dot{\tilde{x}} = -2\tilde{x} - 2x_r - \dot{x}_r + (2\tilde{x} + 1.5 + 2x_r)(\tilde{x} + x_r + \Gamma_s)$$
$$- (\tilde{x} + x_r + \Gamma_s)^2 \qquad (93)$$

Design Γ_s as

$$\Gamma_s = -\tilde{x} - x_r + \dot{x}_r - A\tilde{x} \qquad (94)$$

where A is the feedback gain. Thus, the resulting closed-loop reduced slow subsystem is

$$\dot{\tilde{x}} = -(2 - 2\dot{x}_r + 2Ax_r + 1.5A - 2A\dot{x}_r)\tilde{x} + (-A^2 - 2A)\tilde{x}^2$$
$$+ (-2x_r + 0.5\dot{x}_r + 2x_r\dot{x}_r - \dot{x}_r^2) \qquad (95)$$

where A is the feedback gain. Comparing Eq. (95) with Eq. (50),

$$\mathbf{F}_s(.) = (2 - 2\dot{x}_r + 2Ax_r + 1.5A - 2A\dot{x}_r)\tilde{x} - (-2x_r + 0.5\dot{x}_r$$
$$+ 2x_r\dot{x}_r - \dot{x}_r^2)$$
$$\mathbf{G}_s(.) = (-A^2 - 2A)\tilde{x}^2 \qquad (96)$$

Note that this control only guarantees bounded tracking for the slow subsystem. To implement the control laws, substitute for Γ_s from Eq. (94) into Eq. (82):

$$\phi = \dot{x}_r - A\tilde{x} \qquad (97)$$

and use Eqs. (85) and (97):

$$\Gamma_f = (-A^2 - A)\tilde{x}^2 + \tilde{x}(\dot{x}_r + 2A\dot{x}_r - Ax_r) + 2A\tilde{x}\tilde{z} - \tilde{z}(2\dot{x}_r + A_f)$$
$$- \dot{x}_r^2 + x_r\dot{x}_r \qquad (98)$$

Recall that these controllers are designed using the reduced-order subsystems $\tilde{z} = z - \phi(.)$, where $\phi(.)$ is given by Eq. (97). Then, the control laws Γ_s and Γ_f can be expressed in original coordinates as

$$\Gamma_s = -x + \dot{x}_r - A(x - x_r) \qquad (99)$$

$$\Gamma_f = (-A^2 - A)(x - x_r)^2 + (x - x_r)(\dot{x}_r + 2A\dot{x}_r - Ax_r)$$
$$+ 2A(x - x_r)[z - \phi(.)] - [z - \phi(.)](2\dot{x}_r + A_f) - \dot{x}_r^2 + x_r\dot{x}_r \quad (100)$$

Using the manifold condition equation (83),

$$(M\phi)(\tilde{x}, x_r, \dot{x}_r, \Gamma_s) = \epsilon\frac{\partial\phi}{\partial t} + \epsilon\frac{\partial\phi}{\partial\tilde{x}}\{-\mathbf{F}_s(\tilde{x}, x_r, \dot{x}_r) + \mathbf{G}_s(\tilde{x})\} \quad (101)$$

Thus, by the choice of controls Γ_s and Γ_f, $\mathcal{O}[C(\epsilon = 0, \tilde{x}, x_r, \dot{x}_r)] = 0$.

2. Results and Discussion

Case 1A: Controller performance for tracking a continuously time-varying sine wave of $0.2\sin(0.2t)$ is presented in Fig. 4. The feedback gains chosen are $A = 3$ and $A_f = 1$. The domains of the errors are $D_x = [-0.3 \quad 0.3]$ and $D_z = [-1.5 \quad 1.5]$. Several constants in Assumptions B–I are computed as $\alpha_1 = 1$, $b_1 = 0.26$, $\beta_1 = 1.4$, $\beta_2 = 30$, $\beta_3 = 0$, $\beta_4 = 0.686$, $\alpha_2 = 1$, $\beta_5 = 1.96$, $\beta_6 = 250$, $\beta_7 = 0.5096$, $\beta_8 = 3.778$, and $\beta_9 = 250$. These values and a choice of $d = 0.3$ results in $\epsilon^* = 2000 \gg 1$. From the simulation results, it is seen that the system response is bounded for all time. Additionally, for simulations with $\epsilon = 0.2$, the bounds
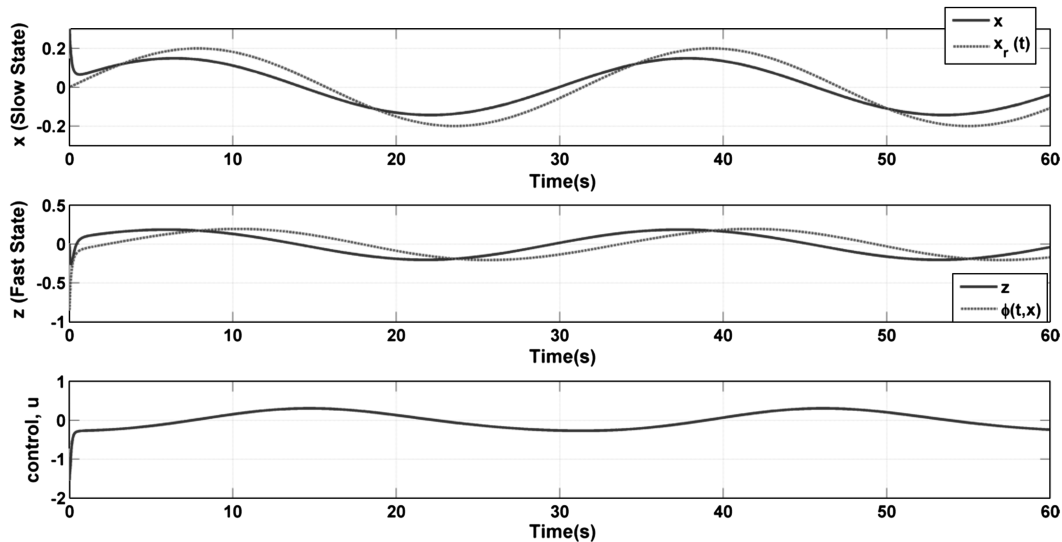

Fig. 4   Case 1A: kinetic slow state compared with specified sine-wave reference, and fast state compared with manifold approximation and computed control.
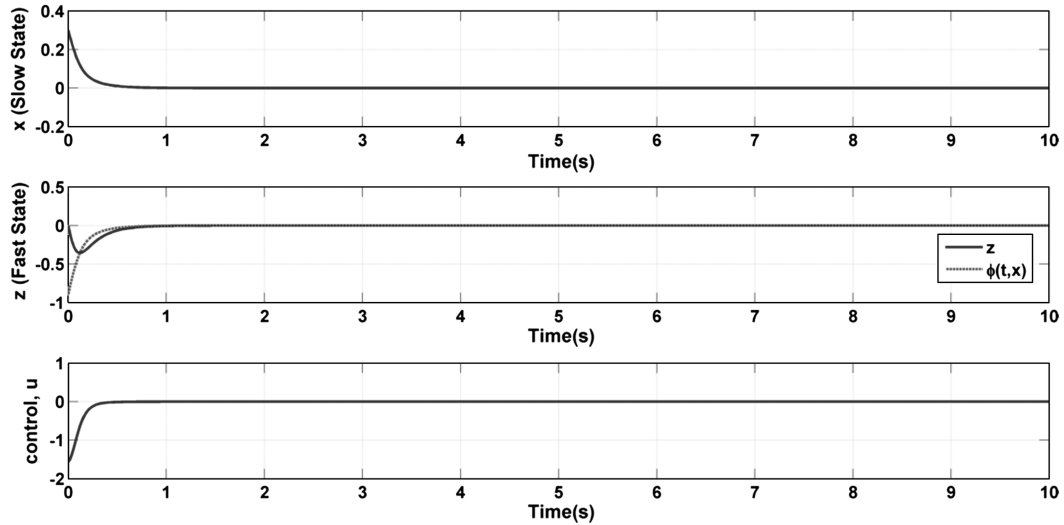
Fig. 5   Case 1B: kinetic slow state, fast state compared with manifold approximation and computed control (regulator problem).

Fig. 6   F/A-18A Hornet external physical characteristics.

$\tilde{x}_b = 0.0818$ and $\tilde{z}_b = 4.701$, and the control is bounded for all time. The initial overshoot may be avoided by adding actuator dynamics and adjusting the feedback gains. Note that the fast state response remains close to its approximation $\phi(t, x)$.

Case 1B: This case simulates the regulator problem with $x_r = 0$ and $\dot{x}_r(t) = 0$. The control laws are the same as derived in Eqs. (99) and (100). The constants $b_1 = 0$, $\beta_4 = 0$, $\beta_7 = 0$, and $\beta_8 = 0$, while the other constants have the same values as in Case 1A. With the choice of $d = 0$, $\epsilon_s^* = 1000 \gg 1$. The results are presented in Fig. 5, which shows that the system asymptotically settles down to the origin.

C.   Lateral/Directional Maneuver for F/A-18A Hornet Aircraft

The complete nonlinear dynamic model in the stability axes is represented by the nine states $(M, \alpha, \beta, p, q, r, \phi, \theta, \psi)$ and four controls $(\eta, \delta_e, \delta_a, \delta_r)$. For this example, $[M, \alpha, \beta, \phi, \theta, \psi]^T$ comprise the slow states, and the angular rates $[p, q, r]^T$ comprise the fast states. The aerodynamic database for the symmetric F/A-18A Hornet (seen in Fig. 6) is used [30]. The aerodynamic coefficients are given as analytical functions of the sideslip angle, angle of attack, angular rates, and the control surface deflections. Considering the number of controls available, only three of the six slow states can be controlled. Throttle is maintained constant at $\eta = 0.523$ and is not used as a control. This is a result of using dynamic inversion [31]. The control objective is to perform a 45 deg turn at or near zero sideslip angle while tracking a specified angle-of-attack profile. Pitch attitude angle θ and bank angle ϕ are left uncontrolled.

1.   Controller Design

The control laws are developed according to the theory developed in the previous sections. For brevity, only the equations required for

incorporating the control law in the simulation are presented here. Since the aircraft equations of motion are highly coupled, the first step is to transform them into slow and fast sets. Let $\mathbf{x} = [\alpha, \beta, \psi]^T$ represent the subset of the slow states and $\mathbf{u} = [\delta_e, \delta_a, \delta_r]^T$ represent the control variables,

$$\dot{\mathbf{x}} = \underbrace{\mathbf{f_1}(\mathbf{x}, M, \theta, \phi) + \mathbf{f_2}(\mathbf{x}, \theta, \phi)\mathbf{z}}_{\mathbf{f}(.)} + \mathbf{g}(\mathbf{x}, M)\mathbf{u} \qquad (102)$$

$$\epsilon\dot{\mathbf{z}} = \underbrace{l(\mathbf{z}) + l(\mathbf{x}, M) + l(\mathbf{x}, M)\mathbf{z}\mathbf{3}\mathbf{2}\mathbf{1}}_{l(.)} + \mathbf{k}(\mathbf{x}, M)\mathbf{u} \qquad (103)$$

The parameter ϵ is introduced on the left-hand side of Eq. (103) to indicate the time-scale difference between body-axis angular rates and the other states [14]. In the translational equations of motion, functions such as gravitational forces and aerodynamic forces due to angle of attack and sideslip angle are collectively represented as $\mathbf{f_1}(\mathbf{x}, M, \theta, \phi)$. Terms in the translational equations of motion due to the cross products between the angular rates and the slow states are labeled $\mathbf{f_2}(\mathbf{x}, \theta, \phi)\mathbf{z}$. The remaining terms in the slow state equations are the control effectiveness terms labeled $\mathbf{g}(\mathbf{x}, M)$. The nonlinearity in the fast dynamics due to the cross product between the angular rates is represented by $l(\mathbf{z})\mathbf{1}$. The aerodynamic moment terms that depend solely upon the slow state are denoted as $l(\mathbf{x}, M)\mathbf{2}$, and the aerodynamic moment terms that depend linearly on the angular rates are denoted as $l(\mathbf{x})\mathbf{3}$. The term $\mathbf{k}(\mathbf{x}, M)$ is the control effectiveness term in the angular rate dynamics. The exact form of these functions is derived in the Appendix. Define the errors $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_r$ and $\tilde{\mathbf{z}} = \mathbf{z} - \mathbf{h}(\tilde{\mathbf{x}}, \epsilon, \mathbf{x}_r, \dot{\mathbf{x}}_r, M)$ and transform Eqs. (102) and (103) into error coordinates equivalent to Eqs. (39) and (40):

$$\dot{\tilde{\mathbf{x}}} = \underbrace{\mathbf{f_1}(\tilde{\mathbf{x}} + \mathbf{x}_r, M, \theta, \phi) + \mathbf{f_2}(\tilde{\mathbf{x}} + \mathbf{x}_r, \theta, \phi)[\tilde{\mathbf{z}} + \mathbf{h}(.)]}_{\mathbf{f}(.)}$$

$$+ \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}, M)[\Gamma_\mathbf{s} + \Gamma_\mathbf{f}] - \dot{\mathbf{x}}_r \qquad (104)$$

$$\epsilon\dot{\tilde{\mathbf{z}}} = \underbrace{l[\tilde{\mathbf{z}} + \mathbf{h}(.)] + l(\tilde{\mathbf{x}} + \mathbf{x}_r, M) + l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)[\tilde{\mathbf{z}} + \mathbf{h}(.)]321}_{l(.)}$$

$$+ \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)[\Gamma_\mathbf{s} + \Gamma_\mathbf{f}] - \epsilon\frac{\partial \mathbf{h}}{\partial t} - \epsilon\frac{\partial \mathbf{h}}{\partial \tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} - \epsilon\frac{\partial \mathbf{h}}{\partial M}\dot{M} \quad (105)$$

Note that, for an aircraft example, the manifold will also be a function of Mach number. Let $\Phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_\mathbf{s}, M)$ be the approximate manifold. Then, Eq. (38) expresses the manifold condition. In this case, select

$$\Phi(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_\mathbf{s}) = -l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)[l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)$$

$$+ \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\Gamma_\mathbf{s}]23 - 1 \qquad (106)$$

such that

$$(M\Phi)(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_\mathbf{s}) = \epsilon\frac{\partial \Phi}{\partial t} + \epsilon\frac{\partial \Phi}{\partial \tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} + \epsilon\frac{\partial \Phi}{\partial M}\dot{M} - l[\Phi(.)]$$

$$- \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\Gamma_\mathbf{f}1 \qquad (107)$$

To design Γ_f, develop the reduced fast subsystem,

$$\tilde{\mathbf{x}}' = \mathbf{0} \qquad (108)$$

$$\tilde{\mathbf{z}}' = l[\tilde{\mathbf{z}} + \Phi(.)] + l(\tilde{\mathbf{x}} + \mathbf{x}_r, M) + l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)[\tilde{\mathbf{z}} + \Phi(.)]$$

$$+ \mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)[\Gamma_\mathbf{s} + \Gamma_\mathbf{f}]321 \qquad (109)$$

Using dynamic inversion and Eq. (106), design

$$\Gamma_\mathbf{f} = \mathbf{k}^{-1}(\mathbf{x} + \mathbf{x}_r, M)\{-A_f\tilde{\mathbf{z}} - l[\tilde{\mathbf{z}} + \Phi(.)] - l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\tilde{\mathbf{z}}\}31$$

$$(110)$$

where A_f is the chosen feedback gain. Then, the closed-loop reduced subsystem becomes

$$\tilde{\mathbf{z}}' = -A_f\tilde{\mathbf{z}} \qquad (111)$$

Comparing with Eq. (48),

$$\mathbf{L_f}(.) = A_f\tilde{\mathbf{z}} \qquad (112)$$

$$\mathbf{K_f}(.) = 0 \qquad (113)$$

Similarly, develop the reduced-order slow subsystem,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{f_1}(\tilde{\mathbf{x}} + \mathbf{x}_r, M, \theta, \phi) - \mathbf{f_2}(\tilde{\mathbf{x}} + \mathbf{x}_r, \theta, \phi)l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)l(\tilde{\mathbf{x}}$$

$$+ \mathbf{x}_r, M) - \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\mathbf{k}^{-1}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)l(\Phi) - \dot{\mathbf{x}}_r$$

$$+ [-\mathbf{f_2}(\tilde{\mathbf{x}} + \mathbf{x}_r, \theta, \phi)l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)^{-1}\mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)$$

$$+ \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)]\Gamma_s3123 - 1 \qquad (114)$$

Then, the control law for the reduced slow subsystem is computed as

$$\Gamma_\mathbf{s} = \mathbf{B}^{-1}\{-A\tilde{\mathbf{x}} + \dot{\mathbf{x}}_r\} + \mathbf{B}^{-1}\{-\mathbf{f_1}(\tilde{\mathbf{x}} + \mathbf{x}_r, M, \theta, \phi)$$

$$+ \mathbf{f_2}(\tilde{\mathbf{x}} + \mathbf{x}_r, \theta, \phi)l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\}23 - 1 \quad (115)$$

where

$$\mathbf{B} = [-\mathbf{f_2}(\tilde{\mathbf{x}} + \mathbf{x}_r, \theta, \phi)l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)^{-1}\mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)$$

$$+ \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)]3$$

A is the feedback gain, and the resulting closed-loop system is

$$\dot{\tilde{\mathbf{x}}} = -A\tilde{\mathbf{x}} - \mathbf{g}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)\mathbf{k}^{-1}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)l[\Phi(.)]1 \qquad (116)$$

where $\Phi(.)$ is obtained from Eq. (106). Note by the choice of Γ_f, Eq. (107) becomes

$$(M\Phi)(\tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r, \Gamma_\mathbf{s}) = \epsilon\frac{\partial \Phi}{\partial t} + \epsilon\frac{\partial \Phi}{\partial \tilde{\mathbf{x}}}\dot{\tilde{\mathbf{x}}} + \epsilon\frac{\partial \Phi}{\partial M}\dot{M} \qquad (117)$$

and thus $\mathcal{O}[C(\epsilon = 0, \tilde{\mathbf{x}}, \mathbf{x}_r, \dot{\mathbf{x}}_r)] = 0$. Furthermore, since the aerodynamic moments are a function of the angular rates, matrix $l(\tilde{\mathbf{x}} + \mathbf{x}_r, M)3$ is full rank. The control effectiveness terms $\mathbf{k}(\tilde{\mathbf{x}} + \mathbf{x}_r, M)$ represent the aerodynamic moment coefficients due to control effector deflections, which are nonzero.

Remark 8: The aircraft example assumes that the Mach number, pitch attitude angle, and bank angle are input stabilizable. Although the angular rates are bounded by the reference trajectory, the Euler angles remain bounded through the exact kinematic relationships. Additionally, since the angle of attack is being tracked and thrust remains constant, the Mach number remains bounded.

2. Results and Discussion

Case 2: The specified maneuver is a 45 deg turn at or near zero sideslip angle while simultaneously tracking a step input in the angle of
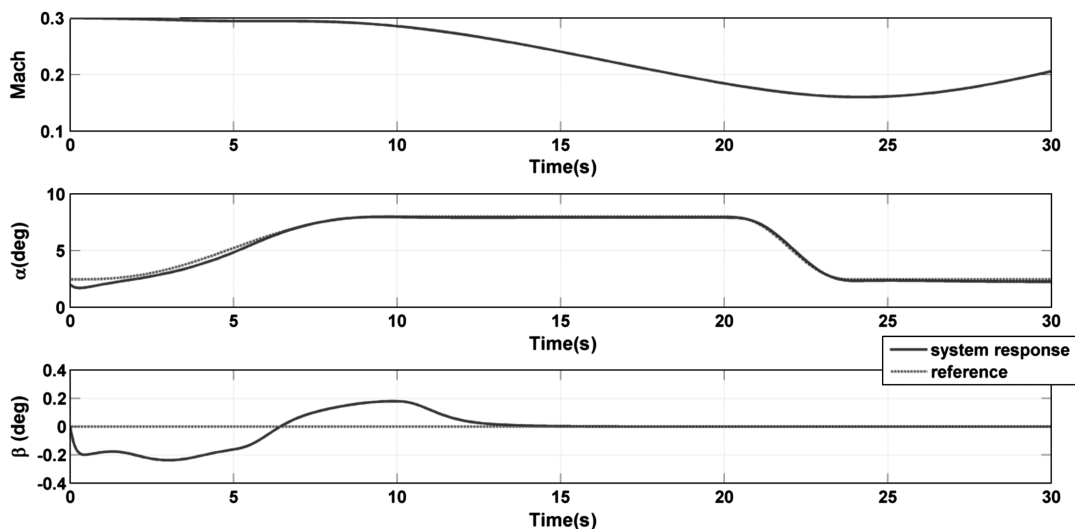

Fig. 7 Case 2: F/A-18A Mach number, angle of attack, and sideslip angle responses; 0.3/20k.
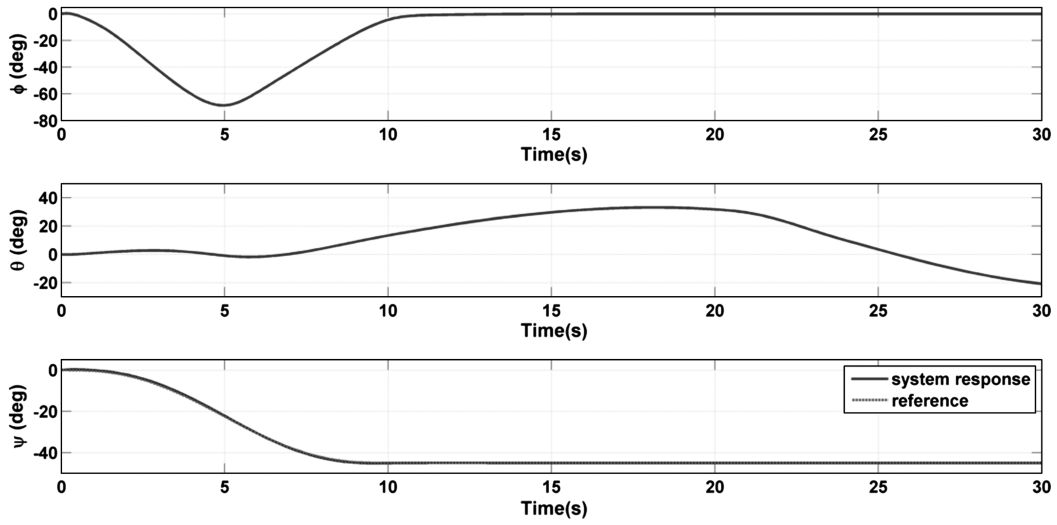
Fig. 8   Case 2: F/A-18A kinematic angle responses; 0.3/20k.
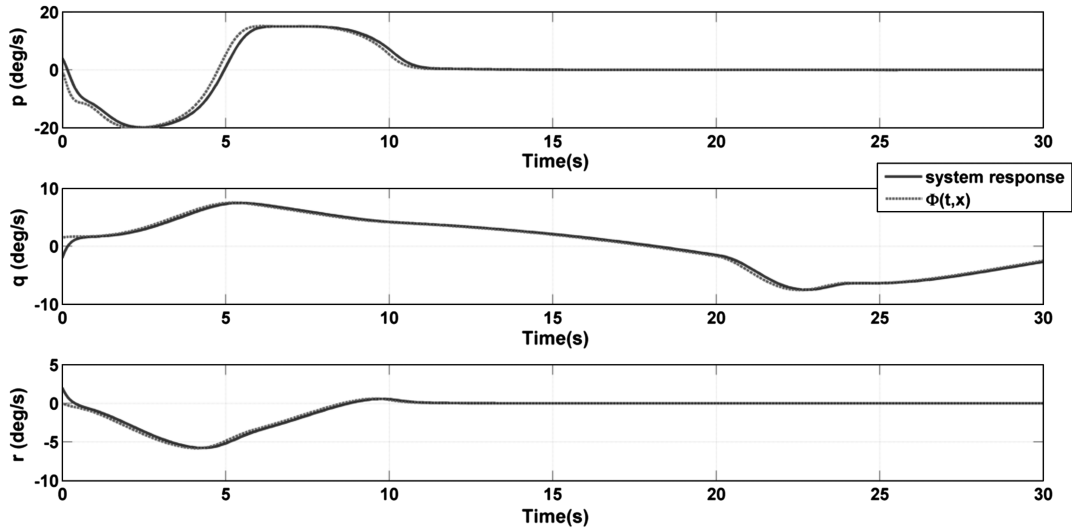

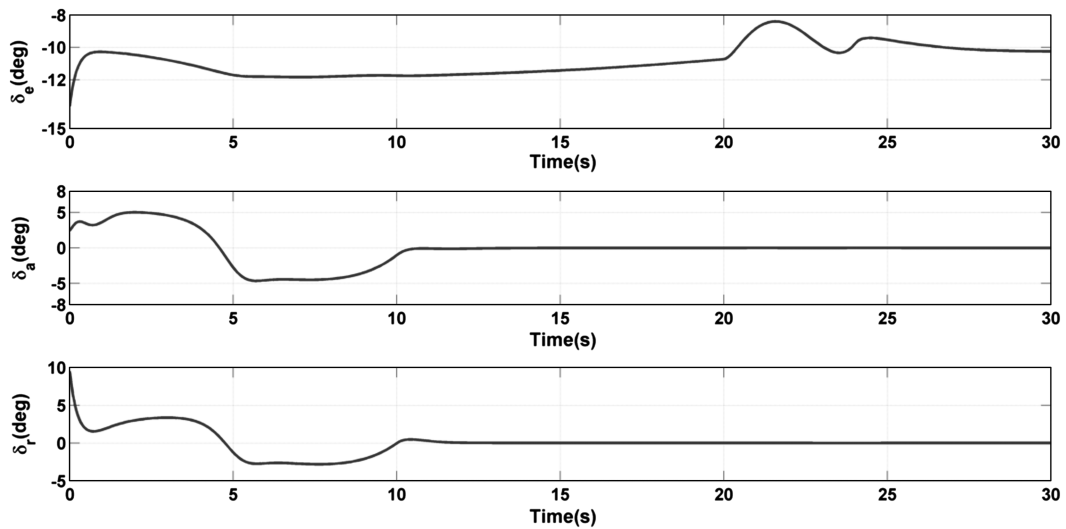Fig. 9   Case 2: F/A-18A angular rate responses; 0.3/20k.

Fig. 10   Case 2: F/A-18A control responses; 0.3/20k.

attack. The flight condition is Mach 0.3 at 20,000 ft altitude (0.3/20k). The trim and initial conditions are $\alpha(1) = 2$ deg, $p(0) = 4$ deg/s, $q(0) = -2$ deg/s, and $r(0) = 2$ deg/s. The feedback gain matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_f = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad (118)$$

Theorem 1 guarantees the existence of the bound ϵ^*, but the nonlinearity of this example restricts its analytical computation. Note also that, for an aircraft, the parameter ϵ is normally only introduced in the modeling stage to take advantage of the presence of different time scales in the system. In reality, this parameter is a function of the flight condition and is difficult to quantify. Thus, it is advantageous to derive and implement controllers that do not require knowledge of this parameter.

Figures 7–10 evaluate control law performance for the specified maneuver. After initial transients settle out, the angle of attack, sideslip angle, and heading angle states closely track the reference. The angle-of-attack error is within ± 0.2 deg, and the sideslip angle tracking error is within ± 0.2 deg throughout the maneuver. The heading angle is maintained within ± 0.25 deg. Close tracking of the slow states implies that the fast states are successfully being driven onto the approximate manifold, as is seen in Fig. 9. The angular rates are smooth, and errors are within ± 2 deg/s. The control surface deflections are within bounds and generate the desired nonzero angular rates. The uncontrolled states $M, \theta,$ and ϕ are well behaved and remain bounded throughout the maneuver.

VII. Conclusions

A control formulation for tracking the slow states of a general class of nonlinear singularly perturbed systems was developed based upon the study of its geometric constructs. For a given set of nonlinear algebraic equations, an approximate analytical form of the system manifold was computed. Control laws for each of the subsystems and boundedness of closed-loop signals was demonstrated with a composite Lyapunov function approach, and asymptotic stabilization was shown for the general class of nonlinear singularly perturbed systems. Controller performance was demonstrated through numerical simulation for two nonlinear examples.

Based upon the results presented in the paper, tracking error for the nonlinear planar example was demonstrated to remain within $|0.08|$ at all times, as predicted by the analytically computed bound. It was also shown that, for all values of ϵ, the controller maintains bounded stability and the asymptotic convergence of the errors to origin for the regulator problem. Nonlinear simulations of an F/A-18A Hornet demonstrate that the controller is capable of closely tracking heading, sideslip angle, and angle of attack. The angular rates were within bounds and seen to track the desired manifold approximations well. Even though the Mach number, bank angle, and pitch attitude angle were not controlled, their magnitudes remained bounded as expected. The aircraft example demonstrates the advantage of developing controllers independent of the scalar perturbation parameter ϵ.

Appendix

The nonlinear mathematical model of the aircraft is represented by the following dynamic and kinematic equations:

$$\dot{M} = \frac{1}{mv_s}\left[T_m \eta \cos\alpha \cos\beta - \frac{1}{2} C_D(\alpha, q, \delta e)\rho v_s^2 M^2 S - mg\sin\gamma \right]$$
$$(A1)$$

$$\dot{\alpha} = q - \frac{1}{\cos\beta}\{(p\cos\alpha + r\sin\alpha)\sin\beta\}$$
$$+ \frac{1}{\cos\beta}\left\{ \frac{1}{mv_s M}\left[T_m \eta \sin\alpha + \frac{1}{2}C_L(\alpha, q, \delta e)\rho v_s^2 M^2 S \right.\right.$$
$$\left.\left. - mg\cos\mu\cos\gamma \right]\right\} \qquad (A2)$$

$$\dot{\beta} = \frac{1}{mv_s M}\left[-T_m\eta\cos\alpha\sin\beta + \frac{1}{2}C_Y(\beta, p, r, \delta e, \delta a, \delta r)\rho v_s^2 M^2 S \right.$$
$$\left. + mg\sin\mu\cos\gamma \right] + (p\sin\alpha - r\cos\alpha) \qquad (A3)$$

$$\dot{p} = \frac{I_y - I_z}{I_x}qr + \frac{1}{2I_x}\rho v_s^2 M^2 SbC_l(\beta, p, r, \delta e, \delta a, \delta r) \qquad (A4)$$

$$\dot{q} = \frac{I_z - I_x}{I_y}pr + \frac{1}{2I_y}\rho v_s^2 M^2 ScC_m(\alpha, q, \delta e) \qquad (A5)$$

$$\dot{r} = \frac{I_x - I_y}{I_z}pq + \frac{1}{2I_z}\rho v_s^2 M^2 SbC_n(\beta, p, r, \delta e, \delta a, \delta r) \qquad (A6)$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \qquad (A7)$$

$$\dot{\theta} = q\cos\phi - r\sin\phi \qquad (A8)$$

$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta \qquad (A9)$$

Wind-axes orientation angles μ and γ are defined as follows:

$$\sin\gamma = \cos\alpha\cos\beta\sin\theta - \sin\beta\sin\phi\cos\theta$$
$$- \sin\alpha\cos\beta\cos\phi\cos\theta \qquad (A10)$$

$$\sin\mu\cos\gamma = \sin\theta\cos\alpha\sin\beta + \sin\phi\cos\theta\cos\beta$$
$$- \sin\alpha\sin\beta\cos\phi\cos\theta \qquad (A11)$$

$$\cos\mu\cos\gamma = \sin\theta\sin\alpha + \cos\alpha\cos\phi\cos\theta \qquad (A12)$$

To write the equations in the form of Eqs. (102) and (103),

$$\mathbf{f}_1(\mathbf{x}, M, \theta, \phi)$$
$$= \begin{bmatrix} -\frac{1}{mv_s M\cos\beta}\left[\frac{1}{2}C_L(\alpha)\rho v_s^2 M^2 S - mg\cos\mu\cos\gamma \right] \\ \frac{1}{mv_s M}\left[\frac{1}{2}C_Y(\beta)\rho v_s^2 M^2 S + mg\sin\mu\cos\gamma \right] \\ 0 \end{bmatrix} \qquad (A13)$$

$$\mathbf{f}_2(\mathbf{x}, \theta, \phi) = \begin{bmatrix} -\cos\alpha\tan\beta & 1 & -\sin\alpha\tan\beta \\ \sin\alpha & 0 & -\cos\alpha \\ 0 & \sec\theta\sin\phi & \cos\phi\sec\theta \end{bmatrix}$$
$$(A14)$$

$\mathbf{g}(\mathbf{x}, M)$

$$= \begin{bmatrix} -\frac{1}{2\cos\beta}\rho(v_s M)^2 S C_{L_{\delta_e}} & 0 & 0 \\ 0 & \frac{1}{2mv_s M}C_{Y_{\delta_a}}\rho v_s^2 M^2 S & \frac{1}{2mv_s M}C_{Y_{\delta_r}}\rho v_s^2 M^2 S \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A15)$$

$$l(\mathbf{z}) = \begin{bmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{bmatrix} \mathbf{1} \qquad (A16)$$

$$l(\mathbf{x}, M) = \begin{bmatrix} \frac{1}{2I_x}\rho v_s^2 M^2 Sb C_l(\beta) \\ \frac{1}{2I_y}\rho v_s^2 M^2 Sc C_m(\alpha) \\ \frac{1}{2I_z}\rho v_s^2 M^2 Sb C_n(\beta) \end{bmatrix} \mathbf{2} \qquad (A17)$$

$l(\mathbf{x}, M)$

$$= \begin{bmatrix} \frac{1}{2I_x}\rho v_s^2 M^2 Sb C_{l_p} & 0 & \frac{1}{2I_x}\rho v_s^2 M^2 Sb C_{l_r} \\ 0 & \frac{1}{2I_y}\rho v_s^2 M^2 Sc C_{m_q} & 0 \\ \frac{1}{2I_z}\rho v_s^2 M^2 Sb C_{n_p} & 0 & \frac{1}{2I_z}\rho v_s^2 M^2 Sb C_{n_r} \end{bmatrix} \mathbf{3}$$

$$(A18)$$

$\mathbf{k}(\mathbf{x}, M)$

$$= \begin{bmatrix} 0 & \frac{1}{2I_x}\rho v_s^2 M^2 Sb C_{l_{\delta_e}} & \frac{1}{2I_x}\rho v_s^2 M^2 Sb C_{l_{\delta_a}} \\ \frac{1}{2I_y}\rho v_s^2 M^2 Sc C_{m_{\delta_e}} & 0 & 0 \\ 0 & \frac{1}{2I_z}\rho v_s^2 M^2 Sb C_{n_{\delta_a}} & \frac{1}{2I_z}\rho v_s^2 M^2 Sb C_{n_{\delta_r}} \end{bmatrix}$$

$$(A19)$$

Acknowledgments

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