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Adaptive Control for Non-Minimum Phase Systems Via Time Scale Separation

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Abstract— **Adaptive control for non-minimum phase systems is a challenging problem. This paper proposes a method of adaptive control for systems that may be both nonlinear and non-minimum phase. This is accomplished by exploiting time scale separation between the internal and external dynamics. The original non-minimum phase control problem is reduced to two minimum phase control problems through a time scale analysis. The resulting adaptive control signals are fused via multiple time scale control techniques. Singular perturbation theory is used to prove the stability and convergence of the full-order system as an extension of the stability and convergence of the two reduced-order systems. The effectiveness of this method is validated on a nonlinear example system.**

I. Introduction

Ioannou and Sun illustrated the significance of the non-minimum phase adaptive control problem. "*The assumption of minimum phase... has often been considered as one of the limitations of adaptive control in general...*" Further, "*The minimum phase assumption is one of the main drawbacks of* [model reference adaptive control] *for the simple reason that the corresponding discrete-time plant of a sampled minimum phase continuous-time plant is often nonminimum* [sic] *phase*" [1, p. 412-413]. Goodwin and Sin showed local stability for Model Reference Adaptive Control (MRAC) on a class of discrete non-minimum phase systems [2]. Johnstone, Shah, and Fisher used control weighting to overcome the non-minimum phase problem [3]. Previously researchers have shown that feedforward terms can make the problem minimum phase [4], [5]. Some model-free adaptive control methods do not require the non-minimum phase assumption [6], [7], [8], [9], [10], [11]. For a general treatise on adaptive methods for non-minimum phase systems see [12].

Recent research has demonstrated control approaches that are well suited for non-minimum phase systems [13, p. 129-185]. However, systems with model uncertainties remain unaddressed. This paper's primary contribution is a novel method of adaptive control for non-minimum phase systems. It is proven that with this method a system's states converge to a reference model under a condition (see Eq. 9) that can be easily checked. Unlike related work, this method utilizes model reduction to simplify implementation and provide insights into the dynamics of the plant. Additionally, this method is a flexible framework that is generally applicable to a wide class of adaptive control algorithms.

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II. Problem Formulation

This work addresses nonlinear non-minimum phase systems of the form

$$\acute{\boldsymbol{x}} = f_x(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{1a}$$

$$\epsilon_z \acute{\boldsymbol{z}} = f_z(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{1b}$$

$$\boldsymbol{y} = C\boldsymbol{x} \tag{1c}$$

The variables $\boldsymbol{x} \in \mathbb{D}_x^\rho$ and $\boldsymbol{z} \in \mathbb{D}_z^{n-\rho}$ are the state variables. $\boldsymbol{u} \in \mathbb{R}^m$ is the system input and $\boldsymbol{y} \in \mathbb{R}^g$ is the system output. Here, $\mathbb{D}_x^\rho \subseteq \mathbb{R}^\rho$ where the superscript indicates the dimension and the subscript indicates a different subspace that is either the result of a bijection of the set $\mathbb{B}^\rho(r_x) \times \mathbb{B}^{n-\rho}(r_z)$ for $r_x, r_z \in \mathbb{R}_+$ or a subset of the same. The variable $\rho \in \mathbb{N}$ is the sum of the relative degrees of the system outputs such that $n > \rho \geq g$. The variable $t_s \in \mathbb{R}_+$ will represent time. The notation $\acute{(\cdot)}$ is the derivative with respect to the time variable t_s. The constant $\epsilon_z \in \mathbb{R}_+$ is called the time scale separation parameter. By definition $\mathcal{O}(f_z) = \mathcal{O}(f_x) = \mathcal{O}(1)$ for the variable ϵ_z. The order of a function (i.e. the output of the $\mathcal{O}$ operator) is a measure of the rate of change of that function as $\epsilon_z \to 0$. See [13, Appendix A.2] for a more formal definition.

The two primary characterizing features of this form (i.e. Eq. 1) are 1 the time scale properties as indicated by the appearance of the time scale separation parameter ϵ_z and 2 that the output is solely dependant upon $\boldsymbol{x}$ through the matrix $C \in \mathbb{R}^{g \times \rho}$. The variable $\boldsymbol{x}$ is known as the system's *external state* and the variable $\boldsymbol{z}$ is called the system's *internal state* because of their respective relationships with the system outputs. The zero dynamics of a system can be determined by setting $\boldsymbol{y} = \dot{\boldsymbol{y}} = 0$ and solving for the remaining dynamics. Clearly, from Eq. 1 this will be equivalent to the internal dynamics with the external states set to 0. If the zero dynamics are unstable then the system is called *non-minimum phase*. This is a generalization of the concept of zeros for linear systems. Indeed the poles of a linear system's zero dynamics are equivalent to the zeros of the full-order system. Thus this method is valid for both linear and nonlinear systems. The majority of adaptive control literature assumes that the zero dynamics are stable about some equilibrium within the domain $\mathbb{D}_x^\rho \times \mathbb{D}_z^{n-\rho}$ (i.e. minimum phase) (e.g. [1], [14], [15]). This work does not make that assumption.

Remark 1. It is worth noting that a wide class of systems can be transformed into the format given in Eq. 1 using a diffeomorphism. Let this diffeomorphism be called R. The existence and uniqueness of R are worth considering. **Existence:** If the system is affine and the time scale separation parameter is ignored then the transformation R is guaranteed to exist [16, p. 566]. This work is not limited to affine systems, but affine systems are common in adaptive control applications. Thus the existence of R will usually depend upon the time scale separation. In multiple time scale control, it is typically assumed that $0 < \epsilon_z \ll 1$. However, the method proposed herein is effective even if $\epsilon_z > 1$. It is unlikely that ϵ_z will be exactly equal to 1. Thus the transformation R is likely to exist and this method is widely applicable. **Uniqueness:** R is not typically unique [16, p. 523]. So R can be selected to ensure that the system is controllable. Unlike [16] no constraints are placed upon $\boldsymbol{z}$. Thus it is possible for $\epsilon_z \acute{\boldsymbol{z}}$ to be dependent upon the input.

A. Mathematical Notation

Let $\mathcal{L}(\cdot)$ be the Lie derivative operator along the vector field inside the parenthesis. Let the operator $|(\cdot)|_p$ be the l_p norm of a vector with finite dimension. If this operator is applied to a matrix then it is the induced l_p norm of the matrix. If there is no subscript then it is the absolute value of a scalar. Let the operator $\|(\cdot)\|_p$ be the L_p norm over time. If this operator is applied to a vector then it will denote the L_p of each component of the vector. In both cases $p \in [1, \infty)$. However, the same notation will apply to the infinity norm.

III. System Time Scales

The *time scale* of a system state variable is a measure of how quickly that state converges or diverges. The systems considered in this paper will have two-time scales. The fast states will converge on the fast time scale t_f and the slow states will converge on the slow time scale t_s. The time scale separation parameter, often denoted $\epsilon_z = t_s/t_f$, is the ratio between the two time scales. Conversion between fast time and slow time is a change of units. Derivatives with respect to the different time scales are useful. They are $d(\cdot)/dt_s = \acute{(\cdot)}$ and $d(\cdot)/dt_f = \grave{(\cdot)}$. These derivatives are related through the relationship $\grave{(\cdot)} = \epsilon_z \acute{(\cdot)}$.

The class of adaptive control algorithms considered here use time varying control state variables that are described by differential equations (i.e. a reference model and adapting gains). These additional control states are added to the system to create an augmented system. Let $\boldsymbol{x}_m \in \mathbb{D}_x^\rho$ and $\boldsymbol{z}_m \in \mathbb{D}_z^{n-\rho}$ be bounded reference model states. Let $\boldsymbol{r}_x \in \mathbb{R}^l$ and $\boldsymbol{r}_z \in \mathbb{R}^k$ be the inputs to these reference models. The parameters $\hat{\boldsymbol{\theta}}_x \in \mathbb{P}_x^i$ and $\hat{\boldsymbol{\theta}}_z \in \mathbb{P}_z^j$ are adaptive estimates of the bounded true parameters $\boldsymbol{\theta}_x \in \mathbb{P}_x^i$ and $\boldsymbol{\theta}_z \in \mathbb{P}_z^j$ respectively. Similar to $\mathbb{D}$, the set $\mathbb{P}_x^i \subseteq \mathbb{R}^i$ is the result of a bijection of the set $\mathbb{B}^i(r_{\theta x})$ and the set $\mathbb{P}_z^j \subseteq \mathbb{R}^j$ is the result of a bijection of the set $\mathbb{B}^j(r_{\theta z})$ for $r_{\theta x}, r_{\theta z} \in \mathbb{R}_+$. Eq. 1 is now augmented with unspecified differential equations that describe the time evolution of these control states. The result is

$$\acute{\boldsymbol{x}} = f_x(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{2a}$$

$$\acute{\boldsymbol{x}}_m = q_x(\boldsymbol{x}_m, \boldsymbol{r}_x) \tag{2b}$$

$$\acute{\hat{\boldsymbol{\theta}}}_x = s_x(\boldsymbol{x}, \boldsymbol{x}_m, \boldsymbol{r}_x) \tag{2c}$$

$$\epsilon_z \acute{\boldsymbol{z}} = f_z(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) \tag{2d}$$

$$\epsilon_z \acute{\boldsymbol{z}}_m = q_z(\boldsymbol{z}_m, \boldsymbol{r}_z) \tag{2e}$$

$$\epsilon_z \acute{\hat{\boldsymbol{\theta}}}_z = s_z(\boldsymbol{z}, \boldsymbol{z}_m, \boldsymbol{r}_z) \tag{2f}$$

The output equation is the same for the remainder of this paper so $\boldsymbol{y}$ is dropped. The following assumption is important to the stability analysis.

Assumption 1. The time scale of the reference models and the adaption laws must match the time scale of the subsystem (i.e. internal or external dynamics) to which they are applied. Mathematically this means $\mathcal{O}(q_z) = \mathcal{O}(q_x) = \mathcal{O}(s_z) = \mathcal{O}(s_x) = \mathcal{O}(1)$.

Physically this assumption means that the relative speed of the reference models for the internal and external states is the same as the relative speed for the internal and external dynamics. This makes intuitive sense because the method developed herein relies on time scale separation. If the slow reference model moved too quickly then the slow states would not be able to keep up - or, more precisely, the evolution of the slow states would not be able to be decoupled from the evolution of the fast states using the time scale analysis to follow.

A final transformation will be applied to Eq. 2 so that it is in error coordinates. Let $\boldsymbol{e}_x \triangleq \boldsymbol{x} - \boldsymbol{x}_m$, $\boldsymbol{e}_z \triangleq \boldsymbol{z} - \boldsymbol{z}_m$, $\tilde{\boldsymbol{\theta}}_x \triangleq \hat{\boldsymbol{\theta}}_x - \boldsymbol{\theta}_x$ and $\tilde{\boldsymbol{\theta}}_z \triangleq \hat{\boldsymbol{\theta}}_z - \boldsymbol{\theta}_z$, such that $\boldsymbol{e}_x \in \mathbb{B}^\rho(r_x)$, $\boldsymbol{e}_z \in \mathbb{B}^{n-\rho}(r_z)$, $\tilde{\boldsymbol{\theta}}_x \in \mathbb{B}^i(r_{\theta x})$, and $\tilde{\boldsymbol{\theta}}_z \in \mathbb{B}^j(r_{\theta z})$. Thus

$$\acute{\boldsymbol{e}}_x = f_x(\boldsymbol{e}_x + \boldsymbol{x}_m, \boldsymbol{e}_z + \boldsymbol{z}_m, \boldsymbol{u}) - \acute{\boldsymbol{x}}_m \tag{3a}$$

$$\acute{\tilde{\boldsymbol{\theta}}}_x = s_x(\boldsymbol{e}_x + \boldsymbol{x}_m, \boldsymbol{x}_m, \boldsymbol{r}_x) - \acute{\boldsymbol{\theta}}_x \tag{3b}$$

$$\epsilon_z \acute{\boldsymbol{e}}_z = f_z(\boldsymbol{e}_x + \boldsymbol{x}_m, \boldsymbol{e}_z + \boldsymbol{z}_m, \boldsymbol{u}) - \epsilon_z \acute{\boldsymbol{z}}_m \tag{3c}$$

$$\epsilon_z \acute{\tilde{\boldsymbol{\theta}}}_z = s_z(\boldsymbol{e}_z + \boldsymbol{z}_m, \boldsymbol{z}_m, \boldsymbol{r}_z) - \epsilon_z \acute{\boldsymbol{\theta}}_z \tag{3d}$$

To simplify notation let $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{e}_x^T & \boldsymbol{x}_m^T & \tilde{\boldsymbol{\theta}}_x^T \end{bmatrix}^T$ and $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{e}_z^T & \boldsymbol{z}_m^T & \tilde{\boldsymbol{\theta}}_z^T \end{bmatrix}^T$. This allows Eq. 3 to be rewritten compactly as

$$\acute{\boldsymbol{\chi}} = f_\chi(\boldsymbol{\chi}, \boldsymbol{\zeta}, \boldsymbol{u}, \acute{\boldsymbol{x}}_m, \acute{\boldsymbol{\theta}}_x) \tag{4a}$$

$$\epsilon_z \acute{\boldsymbol{\zeta}} = f_\zeta(\boldsymbol{\chi}, \boldsymbol{\zeta}, \boldsymbol{u}, \epsilon_z \acute{\boldsymbol{z}}_m, \epsilon_z \acute{\boldsymbol{\theta}}_z) \tag{4b}$$

It is now assumed that these are sufficiently smooth bounded functions within the domain of interest.

Assumption 2. The functions f_ζ and f_x are sufficiently smooth and defined such that $\boldsymbol{r}_z, \boldsymbol{r}_x, \boldsymbol{\zeta}, \boldsymbol{x} \in L_\infty \implies \grave{\boldsymbol{\zeta}}, \acute{\boldsymbol{x}} \in L_\infty$

In other words, Assumption 2 requires that the equations of motion be continuously differentiable as many times as necessary and have no singularities. This assumption will be used in the stability analysis to ensure the applicability of Barbalet's Lemma [1, Lemma 3.2.5].

As stated previously, the general premise of this approach is to drive the fast states to a stable equilibrium and then control the slow states. Conceptually this separates the system into two reduced subsystems - one fast and one slow. In the reduced fast subsystem the slow states are assumed to be stationary because they evolve so slowly that their evolution has a relatively little effect on the fast states. In the reduced slow subsystem the fast states are assumed to have already reached their steady-state stable equilibrium trajectory. This trajectory is known as the *manifold*. The subscript 1 will be used to represent a solution to the fast subsystem and the subscript 0 will be used to represent a solution to the slow subsystem.

Assumption 3. The fast reference model is selected to be the manifold ($\boldsymbol{z}_m \triangleq \boldsymbol{z}_0$).

The time scale analysis to follow will assume that the fast states have reached the manifold by the time the slow states begin to evolve. Thus the fast states must be asymptotically stable to the manifold. That is the purpose of Assumption 3. Now the reduced subsystems will be studied. The reduced subsystems are asymptotic solutions to the closed-loop system. This requires an approximation of the relative time scales of the internal and external dynamics. There are three possible cases.

A. Case 1: $\epsilon_z < 1$

This is the case where the internal dynamics are faster than the external dynamics. The reduced slow subsystem can then be found by taking the limit as $\epsilon_z \to 0$

$$\acute{\boldsymbol{\chi}}_0 = f_\chi(\boldsymbol{\chi}_0, \boldsymbol{\zeta}_0, \boldsymbol{u}_0, \acute{\boldsymbol{x}}_{m0}, \acute{\boldsymbol{\theta}}_{x0}) \tag{5a}$$

$$0 = f_\zeta(\boldsymbol{\chi}_0, \boldsymbol{\zeta}_0, \boldsymbol{u}_0, 0, 0) \tag{5b}$$

Recalling the relationship $\grave{(\cdot)} = \epsilon_z \acute{(\cdot)}$, applying it to Eq. 4, and again taking the limit as $\epsilon_z \to 0$ gives the reduced fast subsystem

$$\grave{\boldsymbol{\chi}}_1 = 0 \tag{6a}$$

$$\grave{\boldsymbol{\zeta}}_1 = f_\zeta(\boldsymbol{\chi}_1, \boldsymbol{\zeta}_1, \boldsymbol{u}_1, \grave{\boldsymbol{z}}_{m1}, \grave{\boldsymbol{\theta}}_{z1}) \tag{6b}$$

B. Case 2: $\epsilon_z > 1$

This is the case where the external dynamics are faster than the internal dynamics. Taking the limit as $\epsilon_z \to 0$ does not accurately represent an asymptotic solution to the system because the fast and slow states are reversed. Let $\epsilon_x \triangleq 1/\epsilon_z$. Using the relationship $\grave{(\cdot)} = \epsilon_z \acute{(\cdot)}$ gives

$$\epsilon_x \grave{\boldsymbol{\chi}} = f_\chi(\boldsymbol{\chi}, \boldsymbol{\zeta}, \boldsymbol{u}, \epsilon_x \grave{\boldsymbol{x}}_m, \epsilon_x \grave{\boldsymbol{\theta}}_x) \tag{7a}$$

$$\grave{\boldsymbol{\zeta}} = f_\zeta(\boldsymbol{\chi}, \boldsymbol{\zeta}, \boldsymbol{u}, \grave{\boldsymbol{z}}_m, \grave{\boldsymbol{\theta}}_z) \tag{7b}$$

Equation 7 can now be used in place of Eq. 4. The reduced subsystems can be found in the same manner as in Case 1.

C. Case 3: $\epsilon_z = 1$

In this case, the internal and external dynamics are in the same time scale. This case is not addressed by this work. However, it is uncommon to encounter a system with ϵ_z exactly equal to 1.

D. Summary

In both Case 1 and Case 2 a reduced slow subsystem and a reduced fast subsystem are identified. Without loss of generality to Case 2, only Case 1 will be considered. It is worth noting that any given point in $\mathbb{D}_x^\rho \times \mathbb{D}_z^{n-\rho}$ is also in the domain of the reduced subsystems with the notable exception that in the reduced slow subsystem the fast states must be on the manifold. Thus the following relationships hold at any given point $\boldsymbol{\chi} = \boldsymbol{\chi}_0 = \boldsymbol{\chi}_1$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}_1$. Importantly, $\boldsymbol{\zeta} \neq \boldsymbol{\zeta}_0$. It follows from Eqs. 4, 5, and 6 that $\acute{\boldsymbol{x}}_m = \acute{\boldsymbol{x}}_{m0}$, and $\grave{\boldsymbol{z}}_m = \grave{\boldsymbol{z}}_{m1}$, and $\acute{\boldsymbol{\theta}}_x = \acute{\boldsymbol{\theta}}_{x0}$, and $\grave{\boldsymbol{\theta}}_z = \grave{\boldsymbol{\theta}}_{z1}$. The subsystem inputs are not necessarily equal to each other or the full-order input. However, all of the multiple time scale control techniques considered in this paper will use $\boldsymbol{u} \triangleq \boldsymbol{u}_1$. Using these relationships it follows from Eqs. 4, 5, and 6 that $\grave{\boldsymbol{\zeta}} = \grave{\boldsymbol{\zeta}}_1$. However, $\acute{\boldsymbol{\chi}} \neq \acute{\boldsymbol{\chi}}_0$. This inequality occurs because f_x is a function of $\boldsymbol{z} \neq \boldsymbol{z}_0$. It also follows from Eqs. 2 and 3 that $\acute{\boldsymbol{x}}_m = \acute{\boldsymbol{x}}_{m0}$ and $\acute{\tilde{\boldsymbol{\theta}}}_x = \acute{\tilde{\boldsymbol{\theta}}}_{x0}$ because q_x and s_x respectively are not functions of any component of $\boldsymbol{\chi}$. These facts are useful in the stability analysis.

IV. Control Formulation

The control framework proposed here is to select separate control signals for the subsystems and then fuse those signals. This section will proceed by showing that if the subsystems are stable, then under certain conditions the full-order system is also stable. Following that proof, several possible fusion methods will be described.

A. Full-Order Stability Contingent Upon reduced-order Stability

Consider the following two Lyapunov functions:

$$V(\boldsymbol{e}_x, \tilde{\boldsymbol{\theta}}_x) : \mathbb{B}^\rho(r_x) \times \mathbb{B}^i(r_{\theta x}) \to \mathbb{R}_{\geq 0} \tag{8a}$$

$$W(\boldsymbol{e}_z, \tilde{\boldsymbol{\theta}}_z) : \mathbb{B}^{n-\rho}(r_z) \times \mathbb{B}^j(r_{\theta z}) \to \mathbb{R}_{\geq 0} \tag{8b}$$

These Lyapunov functions are positive definite functions of class C^1 (i.e. the function and its derivative are continuous) where $V(0,0) = W(0,0) = 0$.

Assumption 4. The adaptive control for the reduced subsystems (i.e. $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, q_x, q_z, s_x, and s_z) is defined such that V and W are known and exists such that $\mathcal{L}(\acute{\boldsymbol{\chi}}_0)V \leq -\alpha_0|\boldsymbol{e}_x|_2^2$ and $\mathcal{L}(\grave{\boldsymbol{\zeta}}_1)W \leq -\alpha_1|\boldsymbol{e}_z|_2^2$ for some $\alpha_0, \alpha_1 \in \mathbb{R}_+$.

Assumption 4 is a formal way of saying, and indeed implies that the reduced subsystems are designed to be stable and $\boldsymbol{e}_{x0}, \boldsymbol{e}_{z1} \to 0$ as $t_f \to \infty$. It is important to note that these conclusions are only valid if the subsystems are not interconnected. Additional work is needed to extend these conclusions to the full-order system because the subsystems are coupled. This is the purpose of Theorem 1.

Theorem 1. *If* $\exists \beta \in \mathbb{R}_{\geq 0}$ *such that*

$$\mathcal{L}(\acute{\boldsymbol{x}} - \acute{\boldsymbol{x}}_0)V \leq \beta|\boldsymbol{e}_x|_2|\boldsymbol{e}_z|_2 \tag{9}$$

Then $\boldsymbol{e}_x, \boldsymbol{e}_z \to 0$ *as* $t \to \infty$.

Proof: The proof for this theorem is similar to the one proposed by [17] for multiple time scale systems. However, it has been significantly altered to account for adaptive control. Define a composite Lyapunov function for the full-order closed-loop system (Eq. 3)

$$\nu = d^* V + dW \tag{10}$$

where $d \in (0,1)$ and $d^* \triangleq (1-d) \in (0,1)$. Differentiating

$$\acute{\nu} = d^* \mathcal{L}(\acute{\boldsymbol{\chi}})V + d\mathcal{L}(\acute{\boldsymbol{\zeta}})W \tag{11}$$

Adding and subtracting $d^* \mathcal{L}(\acute{\boldsymbol{\chi}}_0)V$ gives

$$\acute{\nu} = d^* \mathcal{L}(\acute{\boldsymbol{\chi}}_0)V + d^* \mathcal{L}(\acute{\boldsymbol{\chi}} - \acute{\boldsymbol{\chi}}_0)V + \frac{d}{\epsilon_z}\mathcal{L}(\grave{\boldsymbol{\zeta}})W \tag{12}$$

Recall that $\grave{\boldsymbol{\zeta}} = \grave{\boldsymbol{\zeta}}_1$ and $\acute{\boldsymbol{\chi}} \neq \acute{\boldsymbol{\chi}}_0$. Also recall that $\acute{\tilde{\boldsymbol{\theta}}}_x = \acute{\tilde{\boldsymbol{\theta}}}_{x0}$ and $\acute{\boldsymbol{x}}_m = \acute{\boldsymbol{x}}_{m0}$. Thus the only component of $\acute{\boldsymbol{\chi}}$ that isn't canceled is $\acute{\boldsymbol{x}}$

$$\acute{\nu} = d^* \mathcal{L}(\acute{\boldsymbol{\chi}}_0)V + \frac{d}{\epsilon_z}\mathcal{L}(\grave{\boldsymbol{\zeta}}_1)W + d^* \mathcal{L}(\acute{\boldsymbol{x}} - \acute{\boldsymbol{x}}_0)V \tag{13}$$

By definition $\mathcal{L}(\acute{\boldsymbol{\chi}}_0)V \leq -\alpha_0|\boldsymbol{e}_x|_2^2$ and $\mathcal{L}(\grave{\boldsymbol{\zeta}}_1)W \leq -\alpha_1|\boldsymbol{e}_z|_2^2$. Substitute these values and the condition from Eq. 9

$$\acute{\nu} \leq -d^*\alpha_0|\boldsymbol{e}_x|_2^2 - \frac{d}{\epsilon_z}\alpha_1|\boldsymbol{e}_z|_2^2 + d^*\beta|\boldsymbol{e}_z|_2|\boldsymbol{e}_x|_2 \tag{14}$$

Let $\boldsymbol{v} = \begin{bmatrix}|\boldsymbol{e}_x|_2 & |\boldsymbol{e}_z|_2\end{bmatrix}^T$. Rearranging gives $\acute{\nu} \leq -\boldsymbol{v}^T K \boldsymbol{v}$ where

$$K = \begin{bmatrix} d^*\alpha_0 & -\frac{1}{2}d^*\beta \\ -\frac{1}{2}d^*\beta & \frac{d}{\epsilon_z}\alpha_1 \end{bmatrix} \tag{15}$$

By Sylvester's Criterion [18] the matrix K is positive definite if and only if the leading principle minors are positive. If $\beta > 0$ then the leading principle minors of K are positive if $0 < d^*\alpha_0$ and $\left(4\alpha_0\alpha_1/\epsilon_z\beta^2 + 1\right)^{-1} < d$. Note that $d \in (0,1)$ is an arbitrary number and all variables in these inequalities are positive. Thus, $\exists d \in (0,1)$ such that both inequalities are simultaneously satisfied. Now the case where $\beta = 0$ must be examined. In this case, it can be seen by inspection that K is positive definite (K is a diagonal matrix with positive diagonal entries). Therefore, in all cases $\acute{\nu} \leq 0$. Thus, via Lyapunov's direct method [1, Theorem 3.4.1] it is known that $\boldsymbol{e}_x, \boldsymbol{e}_z, \tilde{\boldsymbol{\theta}}_x, \tilde{\boldsymbol{\theta}}_z, \acute{\nu} \in L_\infty$. Using Assumption 2, $\acute{\boldsymbol{e}}_x, \acute{\boldsymbol{e}}_z \in L_\infty$. Because K is positive definite and symmetric $\exists \lambda \in \mathbb{R}_+$ such that $\acute{\nu} \leq -\lambda|\boldsymbol{v}|_2^2$. Thus it is known by Lemma 1 that $|\boldsymbol{e}_x|_2, |\boldsymbol{e}_z|_2 \in L_2$ (see the Appendix for proof of the lemmas). Further, Lemma 2 gives $|\boldsymbol{e}_x|_2, |\boldsymbol{e}_z|_2 \in L_1$. Again using Lemma 2 gives $\boldsymbol{e}_x, \boldsymbol{e}_z \in L_2$.

In summary, it has been shown that $\boldsymbol{e}_x, \boldsymbol{e}_z, \grave{\boldsymbol{e}}_x, \grave{\boldsymbol{e}}_z \in L_\infty$ and $\boldsymbol{e}_x, \boldsymbol{e}_z \in L_2$. Thus it is known that $\boldsymbol{e}_x, \boldsymbol{e}_z \to 0$ as $t \to \infty$ by Barbalat's Lemma [1, Lemma 3.2.5]. ■

$\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are the portions of $\boldsymbol{u}$ that remain after the system is converted to the reduced subsystems. Theorem 1 is dependent upon the reduced-order models being stabilized by their respective inputs $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. The control objective now is to select the input $\boldsymbol{u}$ such that this condition is met. In other words, $\boldsymbol{u}$ must be chosen such that both reduced-order systems are simultaneously stabilized. See [13] for more information on each of these control fusion methods.

B. *Composite Control [19]*

Composite control selects the control input to be $\boldsymbol{u} = \boldsymbol{u}_s + \boldsymbol{u}_f$ where $\boldsymbol{u}_0 = \boldsymbol{u}_s$ and $\boldsymbol{u}_1 = \boldsymbol{u}_s + \boldsymbol{u}_f$. This implies that $\boldsymbol{u}_f = 0$ when $\boldsymbol{e}_z = 0$. The engineer first selects $\boldsymbol{u}_s$ so that the reduced slow model is stable. Then the engineer can select $\boldsymbol{u}_f$ such that the reduced fast model is also stable (even in the presence of a nonzero slow input).

C. *Sequential Control [13]*

Sequential control uses the fast states as an input for the slow system. The manifold for the fast states $\boldsymbol{z}_0$ is selected such that the slow states converge to their reference model. Then the input $\boldsymbol{u}$ can be selected to drive the fast states to the desired trajectory $\boldsymbol{z}_0$. Thus sequential control uses $\boldsymbol{u}_0 = \boldsymbol{u}_1 = \boldsymbol{u}$.

D. *Simultaneous Slow and Fast Tracking [20]*

This method of control also uses the control $\boldsymbol{u}_0 = \boldsymbol{u}_1 = \boldsymbol{u}$. However, this control is chosen to inherently stabilize both reduced-order systems simultaneously to an arbitrary trajectory. In practice, this requires the dynamics to be fully actuated. The advantage of this additional constraint is that the slow states and the fast states can both be commanded to any arbitrary trajectory (constrained only by time scales and boundedness). Thus this method is particularly well suited for Case 2 as defined above.

Remark 2. Note that ϵ_z is not required to implement the control. This is advantageous because ϵ_z can be difficult to determine. However, it is required that the engineer have a general estimate for ϵ_z. This allows the engineer to identify if Case 1 or Case 2 is applicable. Finally, a rough approximation of ϵ_z will allow the engineer to design the adaptive laws and reference models so that they evolve on the correct time scale. This ensures that Assumption 1 is met. Beyond these conditions, ϵ_z is allowed to be uncertain.

V. Validation

Consider the nonlinear system

$$\acute{x} = \theta_1\left[\arctan(x) + \pi\right]z + \theta_2\left(\cos(x) + 1\right)u \tag{16a}$$

$$\epsilon\acute{z} = x^2 z - u \tag{16b}$$

$$y = x \tag{16c}$$

where $\theta_1, \theta_2 \in \mathbb{R}_+$ are uncertain model parameters. The zero dynamics are $\acute{z} = \theta_1\pi/\left(2\theta_2\epsilon\right)z$. This is unstable because $\theta_1\pi/\left(2\theta_2\epsilon\right) > 0$. Thus the system is non-minimum phase. The control objective is for the slow states to track the following reference model $\acute{x}_m = -a_{x_m}x_m$ where $a_{x_m} \in \mathbb{R}_+$. The transformation R is the trivial automorphism.

Sequential Control is selected to fuse the control signals for the reduced subsystems. Accordingly, z is treated as the input to the reduced slow subsystem. The manifold z_0 is selected such that the reduced slow subsystem converges to a reference model. Adaptive Nonlinear Dynamic Inversion (ANDI) is selected for this purpose [14, p. 6-12]. Now u must be selected to make the reduced fast subsystem track z_0. The input is selected by inspection to ensure the fast subsystem is Lyapunov sense stable. This yields the following manifold and control signal

$$z_0 = \frac{-a_{x_m} x_m - \hat{\theta}_2 \left(\cos(x) + 1\right) u - k_x e_x}{\hat{\theta}_1 \left(\arctan(x) + \pi\right)} \tag{17a}$$

$$u = x^2 z + k_z e_z \tag{17b}$$

where $k_x, k_z \in \mathbb{R}_+$ are control gains. z_0 appears on both sides of Eq. (17a) so it must be solved. Recall that in the reduced slow subsystem $z = z_0$. Substituting Eq. (17b) into Eq. (17a) and solving for z_0 gives

$$z_0 = \frac{-a_{x_m} x_m - k_x e_x}{\hat{\theta}_1 \left(\arctan(x) + \pi\right) + \hat{\theta}_2 \left(\cos(x) + 1\right) x^2} \tag{18}$$

The adaptation laws are

$$\acute{\hat{\theta}}_1 = \gamma_1 \mathrm{Proj}\left(\hat{\theta}_1, \left(\arctan(x) + \pi\right) z_0 e_x\right) \tag{19a}$$

$$\acute{\hat{\theta}}_2 = \gamma_2 \mathrm{Proj}\left(\hat{\theta}_2, \left(\cos(x) + 1\right) u_0 e_x\right) \tag{19b}$$

where $u_0 = x^2 z_0$ is the input when $z = z_0$ and $\gamma_1, \gamma_2 \in \mathbb{R}_+$ are gains for the adaptation laws.

Consider the following Lyapunov functions for the reduced slow subsystems:

$$V = \frac{1}{2}\left(e_x^2 + \gamma_1^{-1}\tilde{\theta}_1^2 + \gamma_2^{-1}\tilde{\theta}_2^2\right) \tag{20a}$$

$$W = \frac{1}{2} e_z^2 \tag{20b}$$

Per [14, Eq. 1.23] the time derivatives are

$$\acute{V} \leq -k_x e_x^2 \tag{21a}$$

$$\grave{W} = -k_z e_z^2 \tag{21b}$$

Now that the reduced subsystems have been proven independently stable, the interconnection condition must be checked. Assume that the domain is limited to $|x| < x_*$ for some $x_* \in \mathbb{R}_+$ and x is initialized within this region. Note that due to the projection operator, θ_1 and θ_2 are bounded. Let θ_{1*} and θ_{2*} respectively be those bounds. Using these facts it can be shown that

$$\mathcal{L}(\acute{x} - \acute{x}_0)V \leq \left[\theta_{1*}\frac{3}{2}\pi + \theta_{2*}2\left(x_*^2 + k_z\right)\right] e_x e_z \tag{22}$$

Thus the conditions of Theorem 1 are satisfied. By Theorem 1 $e_x, e_z \to 0$ as $t_s \to \infty$.

The values used in this simulation are $\epsilon = 0.1$, $\theta_1 = \theta_2 = 1$, $\gamma_1 = \gamma_2 = 10$. The initial conditions are $x = x_m = 1$ and $z = 0$. The initial error of the adapting parameters $\tilde{\boldsymbol{\theta}}$ is randomly selected from a 0 mean normal distribution with a standard deviation of 10% their true value. The time scale separation parameter ϵ is also simulated to be uncertain. As such its error is sampled from the same distribution. Note that the estimate of the time scale separation parameter is not an adapting parameter. The parameter for the reference model is $a_x = 1$. The control gains are $k_x = 1$ and $k_z = 10$. Figure 1 shows the time evolution of the external dynamics. Figure 2 shows the time evolution of the internal dynamics. Figure 3 shows the time evolution of the adapting parameters.

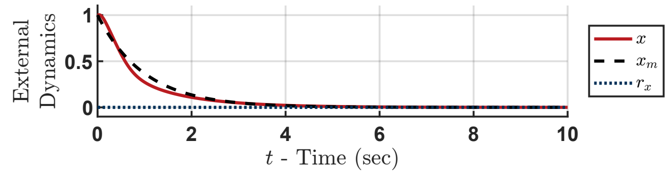

Fig. 1. Evolution of the external dynamics.

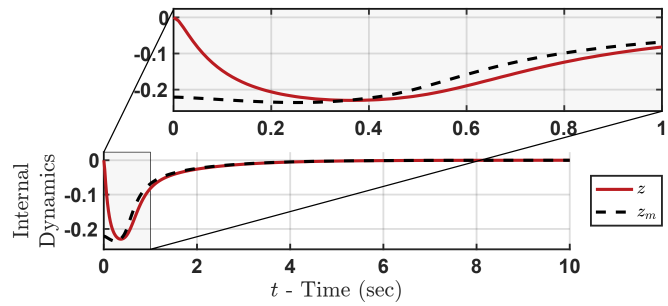

Fig. 2. Evolution of the internal dynamics.

It is worth noting that ANDI alone is incapable of stabilizing this system because it non-minimum phase. Thus the method proposed herein is a significant improvement.

VI. Conclusions

This paper presents a method of adaptive control for a wide class of systems which may be both nonlinear and non-minimum phase. The method requires some time scale separation between the internal and the external dynamics, but a very small time scale separation is permissible. Further, it does not matter if the internal dynamics are faster or the external dynamics are faster. In both cases, the system is a singularly perturbed system suitable for multiple time scale control. A highly general adaptive control architecture has been merged into the system. This means that many different adaptive methods can be used within this framework and allows the given method to take advantage of the most recent adaptive control research. Conditions were given for the stability and convergence of this method. Finally, a validating example was presented. From this work, the following conclusions can be drawn:

1) This method allows adaptive control to be effectively

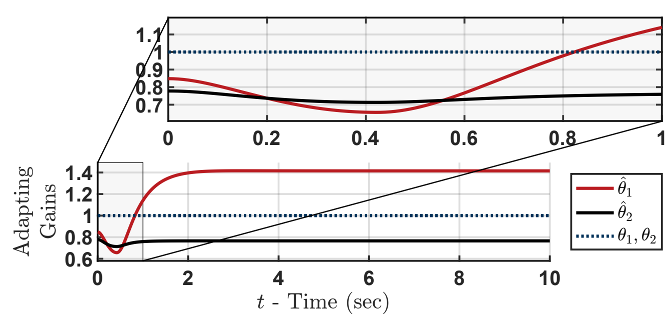

Fig. 3. Evolution of the adapting parameters.

applied to non-minimum phase systems provided there is at least some time scale separation.

2) This method allows significant flexibility because the engineer can choose the adaptive control method, the adaptation laws, and the multiple time scale fusion technique from a wide array of compatible approaches.
3) This method is proven stable and convergent. An engineer can easily ascertain the stability and convergence of a given system and a given control using Theorem 1.

Non-minimum phase systems have been a challenging problem for adaptive control since its inception. The conclusions of this work bring the scientific community one step closer to solving this problem.

APPENDIX

Lemma 1. *Given* $v(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{-}$*,* $\boldsymbol{x}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$*, and* $\alpha \in \mathbb{R}_{+}$ *where* $v \in L_1$ *and* $\boldsymbol{x} \in L_\infty$*. Then* $\forall p \in [1, \infty)$ *it is true that* $v \leq -\alpha |\boldsymbol{x}|_p^p \implies \boldsymbol{x} \in L_p$

Proof: Begin with $v \leq -\alpha |\boldsymbol{x}|_p^p$. The value $|\boldsymbol{x}|_p$ exists and is finite because $\boldsymbol{x} \in L_\infty \implies \boldsymbol{x} \in l_p$ for all p and t. Because both sides of the inequality are negative so the following inequality also holds

$$\|v\|_1 \geq \|\alpha |\boldsymbol{x}|_p^p\|_1 \tag{23}$$

$\|v\|_1/\alpha$ exists and is finite because $v \in L_1$. By Lemma 2 $\boldsymbol{x} \in L_p$. ∎

Lemma 2. *Let* $\boldsymbol{x}(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$*. Then* $\forall p \in [1, \infty)$ *it is true that* $\boldsymbol{x} \in L_p$ *if and only if* $|\boldsymbol{x}|_p^p \in L_1$*.*

Proof: Begin with $\boldsymbol{x} \in L_p$. By the definition of the L_p norm

$$\lim_{\tau \to \infty} \int_0^\tau |x_i|^p dt < \infty \tag{24}$$

where x_i is the i^{th} element of $\boldsymbol{x}$. The sum of a finite quantity of finite numbers is still finite. (Note the inverse of this logic is also true. If the sum of *positive* numbers is finite then each number must also be finite)

$$\lim_{\tau \to \infty} \int_0^\tau \sum_{I=0}^{n} \left[|x_i|^p \right] dt < \infty \tag{25}$$

Raising to the power of 1

$$\lim_{\tau \to \infty} \int_0^\tau \left(\sum_{I=0}^{n} \left[|x_i|^p \right]^{1/p} \right)^p dt < \infty \tag{26}$$

By the definition of the L_p and l_p norms

$$\lim_{\tau \to \infty} \int_0^\tau |\boldsymbol{x}|_p^p dt < \infty \tag{27}$$

Thus $|\boldsymbol{x}|_p^p \in L_1$. Each of these logical steps can be inverted. So, by working backward, it is also true that $|\boldsymbol{x}|_p^p \in L_1 \implies \boldsymbol{x} \in L_p$ ∎

REFERENCES

[1] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice-Hall, 1996.

[2] G. C. Goodwin and K. S. Sin, "Adaptive control of nonminimum phase systems," *Transactions of Automatic Control*, vol. 26, no. 2, pp. 478–483, 1981.

[3] R. M. Johnstone, S. L. Shah, and D. G. Fisher, "An extension of hyperstable adaptive control to non-minimum phase systems," *International Journal of Control*, vol. 31, no. 3, pp. 539–545, 1980.

[4] I. Barkana, "Classical and simple adaptive control for nonminimum phase autopilot design," *Journal of Guidance, Control, and Dynamics*, vol. 28, no. 4, pp. 631–638, 2005.

[5] M. Makoudi and L. Radouane, "A robust model reference adaptive control for non-minimum phase systems with unknown or time-varying delay," *Automatica*, vol. 36, no. 7, pp. 1057–1065, 2000.

[6] D. A. Suarez and R. Lozano, "Adaptive control of nonminimum phase systems subject to unknown bounded disturbances," *Transactions of Automatic Control*, vol. 41, no. 12, pp. 1830–1836, 1996.

[7] M. C. Campi, "Adaptive control of non-minimum phase systems," *International Journal of Adaptive Control and Signal Processing*, vol. 9, no. 2, pp. 137–149, 1995.

[8] M. A. Hersh and M. B. Zarrop, "Stochastic adaptive control of non-minimum phase systems," *Optimal Control Applications and Methods*, vol. 7, no. 2, pp. 153–161, 1986.

[9] L. Praly, S. T. Hung, and D. S. Rhode, "Towards a direct adaptive scheme for a discrete-time control of a minimum phase continuous-time system," in *Conference on Decision and Control*. Institute of Electrical and Electronics Engineers, 1985.

[10] J. B. Hoagg and D. S. Bernstein, "Retrospective cost model reference adaptive control for nonminimum-phase systems," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 6, pp. 1767–1786, 2012.

[11] S. Dai, Z. Ren, and D. S. Bernstein, "Adaptive control of nonminimum-phase systems using shifted laurent series," *International Journal of Control*, vol. 90, no. 3, pp. 407–427, 2017.

[12] D. W. Clarke, "Self-tuning control of nonminimum-phase systems," *Automatica*, vol. 20, no. 5, pp. 501–517, 1984.

[13] A. Narang-Siddarth and J. Valasek, *Nonlinear Multiple Time Scale Systems in Standard and Non-Standard Forms: Analysis & Control*, 1st ed. Society for Industrial and Applied Mathematics, 2014.

[14] D. Famularo, "Observer-based nonlinear dynamic inversion adaptive control with state constraints," Ph.D. dissertation, Texas A&M University, 701 H. R. Bright Building, College Station, TX 77843, September 2017.

[15] N. T. Nguyen, *Model-Reference Adaptive Control A Primer*, ser. Advanced Textbooks in Control and Signal Processing. Springer International Publishing, 2018.

[16] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[17] A. Saberi and H. Khalil, "Stabilization and regulation of nonlinear singularly perturbed systems–composite control," *Institute of Electrical and Electronics Engineers Transactions on Automatic Control*, vol. 30, no. 8, pp. 739–747, August 1985.

[18] G. T. Gilbert, "Positive definite matrices and Sylvester's criterion," *American Mathematical Monthly*, vol. 98, pp. 44–46, January 1991.

[19] P. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control Analysis and Design*, 2nd ed. Society for Industrial and Applied Mathematics, 1999.

[20] A. Narang-Siddarth and J. Valasek, "Global tracking control structures for nonlinear singularly perturbed aircraft systems," Proceedings of the 1st CEAS Specialist Conference on Guidance, Navigation and Control (Euro GNC 2011), Munchen, Germany, 13 April 2011, 2011.