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Global Tracking Control Structures for Nonlinear Singularly Perturbed Aircraft Systems

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Abstract The problem of simultaneous tracking of both fast and slow states for a general class of nonlinear singularly perturbed systems is addressed. A motivating example is an aircraft tracking a prescribed fast moving target, while simultaneously regulating speed and/or one or more kinematic angles. Previous results in the literature have focused on tracking outputs that are a function of the slow states alone. Moreover, global asymptotic tracking has been demonstrated only for a class of nonlinear systems that possess a unique real root for the fast states. For a more general class of nonlinear systems only local tracking results have been proven. In this paper, control laws that accomplish global tracking of both fast and slow states is developed using geometric singular perturbation methods. Global exponential stability is proven using the composite Lyapunov function approach and an upper bound necessary condition for the scalar perturbation parameter is derived. Controller performance is demonstrated through simulation of a combined longitudinal lateral/directional maneuver for a nonlinear, coupled, six degree-of-freedom model of the F/A-18A Hornet. Results presented in the paper show that the controller accomplishes global asymptotic tracking even though the desired reference trajectory requires the aircraft to switch between linear and nonlinear regimes. Asymptotic tracking while keeping all the closed-loop signals bounded and well behaved is also demonstrated. Additionally the controller is independent of the scalar perturbation parameter nor does it require knowledge of it.

1 Introduction

This paper addresses systems that possess both slow and fast dynamics. This multiple time-scale behaviour is either a system characteristic (for example, aircraft

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and flexible beam structures) or arises due to implementation of a fast controller (for example, systems with fast actuators and/or fast sensors). The control objective is to develop a stable tracker for these two time-scale systems that would regulate both slow and fast states simultaneously. The singular perturbation approach[13] has been the foremost technique employed in the literature to examine the behaviour of these two time-scale systems. In this approach, the system dynamics are approximated by two lower-order subsystems. The slow subsystem captures the dominant phenomena assuming that the fast variables evolve infinitely many times faster, and have settled down onto a manifold. The fast subsystem addresses the neglected phenomena, and assumes that the slow variables remain constant. It has been shown that the complete system behaviour can be approximated by the dynamics of the slow subsystem provided the fast subsystem is uniformly asymptotically stable about the manifold [6, 10]. These results of singular perturbation methods have made it the most favourable model-reduction technique in the control literature[14].

The design of nonlinear tracking control laws for the slow variables using singular perturbation methods has received a lot of attention in the past. The typical methodology is to design two separate controllers for each of the two subsystems, and then apply their composite or sum to the full-order system. A tracking control law is designed for the slow subsystem whereas a stabilizing controller is designed for the fast subsystem. This is done to restrict the fast variables onto a manifold. Global asymptotic tracking of the composite control structure is guaranteed only if the manifold is unique. This manifold is the set of fixed points of the fast dynamics expressed as a smooth function of the slow variables and the control inputs; hence it is not always unique. To enforce the uniqueness condition, previous studies in the literature have:

1. Assumed that the system has a unique manifold[4, 8]
2. Considered nonlinear systems that have a unique manifold. This is satisfied by two time-scale systems that are nonlinear in the slow states and linear in the fast states[11]

For a general class of nonlinear systems such as aircraft, approximate approaches that guarantee local stability have been proposed. One approach is to consider the fast variables as control inputs for the slow subsystem. Reference[12] used this approach to design nonlinear flight test trajectories for velocity, angle-of-attack, sideslip angle and altitude by using the fast angular rates as the control variables. This control was augmented with an outer-loop controller that determines the control surface deflections needed to ensure that the angular rates track the output of the inner-loop. More recently the same concept has been employed for the control of generic reentry vehicles[7]. Another approach proposed in Reference[16] considered the general class of nonlinear singularly perturbed systems and computed local approximations of the manifold that helped conclude local stability for the complete system.

All of the approaches discussed above demonstrate slow state tracking either locally or globally by restricting the fast states, and, they address the output tracking problem for two time-scale systems with fast actuators. But for systems whose dynamics inherently possess different time-scales, both the slow and the fast states constitute the output vector. For example, during air combat maneuvering an aircraft is typically required to track a fast moving target while regulating speed (slow variable) and/or one or more kinematic and aerodynamic angles. In this case the fast states cannot be restricted to simply stabilize onto a manifold. The reduced-order approach therefore appears to be inadequate for a general class of output tracking problem. Reference[1] formulated optimal control laws to accomplish fast state tracking using invariant measures for systems with oscillatory fast dynamics.

In this paper, state feedback control laws are developed for a general class of nonlinear singularly perturbed systems to accomplish slow and fast state tracking simultaneously. The paper makes two major contributions. First, the approach developed here employs the reduced-order technique without imposing any assumptions about the fast manifold. This is done by extending the previous work of the authors[16] so as to not require computation of the manifold. Second, global exponential tracking is proved using the composite Lyapunov approach[10]. From the stability analysis it is shown that this approach applies to all classes of singularly perturbed systems, with the global exponential stabilization results of a class of singularly perturbed systems being a special case[3]. Additionally, the technique is independent of the scalar perturbation parameter and an upper bound on this parameter is derived as a necessary condition for stability results to hold. These results are demonstrated by simulation for a nonlinear, coupled, six degree-of-freedom model of the F/A-18A Hornet.

The paper is organized as follows. Section 2 mathematically formulates the control problem and briefly reviews the necessary concepts for model reduction from geometric singular perturbation theory. Control laws and the main results of the paper are detailed in Section 3. Section 4 presents simulation results, and conclusions are presented in Section 5.

2 Problem Formulation and Model Reduction

The following nonlinear singularly perturbed model represents the class of two time-scale dynamical systems addressed in this paper

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z}) + \mathbf{g}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{1}$$

$$\varepsilon\dot{\mathbf{z}} = \mathbf{l}(\mathbf{x},\mathbf{z}) + \mathbf{k}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{2}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^m$ is the vector of slow variables, $\mathbf{z} \in \mathbb{R}^n$ is the vector of fast variables, $\mathbf{u} \in \mathbb{R}^p$ is the input vector and $\mathbf{y} \in \mathbb{R}^{m+n}$ is the output vector. $\varepsilon \in \mathbb{R}^+$ is the singular perturbation parameter that satisfies $0 < \varepsilon << 1$. The vector fields $\mathbf{f}(.), \mathbf{g}(.), \mathbf{l}(.)$ and $\mathbf{k}(.)$ are assumed to be sufficiently smooth and $p \geq (m+n)$. The control objective is to drive the output so as to track sufficiently smooth, bounded, time-varying trajectories, such that $\mathbf{x}(t) \to \mathbf{x}_r(t)$ and $\mathbf{z}(t) \to \mathbf{z}_r(t)$ as $t \to \infty$.

2.1 Reduced-order Modeling

The singularly perturbed model considered in Eqs.1,2 is expressed in the *slow time scale* t. In this time-scale the slow states evolve at an ordinary rate whereas the fast states move at a rate of $O\left(\frac{1}{\varepsilon}\right)$. This system can be equivalently expressed in the *fast time-scale* τ such that the fast states evolve at an ordinary rate and the slow variables move slowly at a rate of $O(\varepsilon)$

$$\mathbf{x}' = \varepsilon\left[\mathbf{f}(\mathbf{x},\mathbf{z}) + \mathbf{g}(\mathbf{x},\mathbf{z})\mathbf{u}\right] \tag{4}$$
$$\mathbf{z}' = \mathbf{l}(\mathbf{x},\mathbf{z}) + \mathbf{k}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{5}$$

where $'$ represents a derivative with respect to $\tau = \frac{t-t_0}{\varepsilon}$ and t_0 is the initial time. Geometric singular perturbation theory[6] examines the behaviour of these singularly perturbed systems by studying the geometric constructs of reduced-order models obtained by substituting $\varepsilon = 0$ in Eqs.1,2 and Eqs.4,5. This results in the *Reduced Slow Subsystem*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{z}) + \mathbf{g}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{6}$$
$$\mathbf{0} = \mathbf{l}(\mathbf{x},\mathbf{z}) + \mathbf{k}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{7}$$

and the *Reduced Fast Subsystem*

$$\mathbf{x}' = \mathbf{0} \tag{8}$$
$$\mathbf{z}' = \mathbf{l}(\mathbf{x},\mathbf{z}) + \mathbf{k}(\mathbf{x},\mathbf{z})\mathbf{u} \tag{9}$$

The reduced slow subsystem is a locally flattened space of the complete system (Eqs.1,2). Since the vector fields were assumed to be sufficiently smooth there exists a smooth diffeomorphism that maps the complete system onto this locally flattened space. The set of points $(\mathbf{x},\mathbf{z},\mathbf{u}) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ is a smooth manifold $\mathscr{M}_0$ of dimension $m+p$ that satisfies the algebraic Eq.7:

$$\mathscr{M}_0 : \mathbf{z} = \mathbf{Z}_0(\mathbf{x},\mathbf{u}) \tag{10}$$

This set of points is identically the fixed points of the reduced fast subsystem (Eq.9). Thus the manifold $\mathscr{M}_0$ is invariant. The flow on this manifold is described by the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x},\mathbf{Z}_0(\mathbf{x},\mathbf{u})\right) + \mathbf{g}\left(\mathbf{x},\mathbf{Z}_0(\mathbf{x},\mathbf{u})\right)\mathbf{u} \tag{11}$$

Fenichel[6] proved that the dynamics of a singularly perturbed system of the form represented in Eqs.1,2 is constrained within $O(\varepsilon)$ of Eq.11 if the reduced fast subsystem is stable about $\mathscr{M}_0$. If the dynamics of Eq.11 are locally asymptotically stable about the manifold, then it can be concluded that the complete system is also locally asymptotically stable. Global asymptotic stability conclusions about the complete system can only be made if the manifold is unique, which is a result from differential topology and center manifold theory [6].

3 Control Formulation and Stability Analysis

The central idea in the development is the following. If the manifold is unique and an asymptotically stable fixed point of the reduced fast subsystem, the complete system follows the dynamics of the reduced slow subsystem globally. Therefore, for a tracking problem addressed in this paper it is desired that this manifold lie exactly on the desired fast state reference for all time. *This condition can be enforced if the nonlinear algebraic set of equations is augmented with a controller that enforces the reference to be the unique manifold*. Additionally, this controller should also be capable of simultaneously driving the slow states to their specified reference. These ideas are mathematically formulated and analyzed in the following subsections.

3.1 Control Law Development

The objective is to augment the two time-scale system with controllers such that the system follows smooth, bounded, time-varying trajectories $[\mathbf{x}_r(t), \mathbf{z}_r(t)]^T$. The first step is to transform the problem into a non-autonomous stabilization control problem. Define the tracking error signals as

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}_r(t) \tag{12}$$
$$\xi(t) = \mathbf{z}(t) - \mathbf{z}_r(t) \tag{13}$$

Substituting Eqs.1,2, the tracking error dynamics are expressed as

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}, \mathbf{z}) + \mathbf{g}(\mathbf{x}, \mathbf{z})\mathbf{u} - \dot{\mathbf{x}}_r \triangleq \mathbf{F}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r, \dot{\mathbf{x}}_r) + \mathbf{G}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r)\mathbf{u} \tag{14}$$
$$\varepsilon\dot{\xi} = \mathbf{l}(\mathbf{x}, \mathbf{z}) + \mathbf{k}(\mathbf{x}, \mathbf{z})\mathbf{u} - \varepsilon\dot{\mathbf{z}}_r \triangleq \mathbf{L}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r, \varepsilon\dot{\mathbf{z}}_r) + \mathbf{K}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r)\mathbf{u} \tag{15}$$

The control law is formulated using the reduced-order models for the complete stabilization problem, which is obtained using the procedure developed in Section 2.
Reduced Slow Subsystem

$$\dot{\mathbf{e}} = \mathbf{F}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r, \dot{\mathbf{x}}_r) + \mathbf{G}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r)\mathbf{u}_0 \tag{16}$$
$$\mathbf{0} = \mathbf{L}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r, \mathbf{0}) + \mathbf{K}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r)\mathbf{u}_0 \tag{17}$$

Reduced Fast Subsystem

$$\mathbf{e}' = \mathbf{0} \tag{18}$$
$$\xi' = \mathbf{L}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r, \mathbf{z}_r') + \mathbf{K}(\mathbf{e}, \xi, \mathbf{x}_r, \mathbf{z}_r)(\mathbf{u}_0 + \mathbf{u}_f) \tag{19}$$

It is known that the fast tracking error ξ will settle onto the manifold that is a function of the error $\mathbf{e}$ and control input $\mathbf{u}$, which may not necessarily be the origin. To steer both errors to the origin, the control input must be designed such that the origin becomes the unique manifold of the reduced slow system (Eqs.16,17). Therefore, the slow controller $\mathbf{u}_0$ is designed to take the form

$$\begin{bmatrix} \mathbf{G}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \\ \mathbf{K}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \end{bmatrix} \mathbf{u}_0 = -\begin{bmatrix} \mathbf{F}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\dot{\mathbf{x}}_r) \\ \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{0}) \end{bmatrix} + \begin{bmatrix} A_{\mathbf{e}}\mathbf{e} \\ A_\xi \xi \end{bmatrix} \tag{20}$$

where $A_{\mathbf{e}}$ and A_ξ specify the desired closed-loop characteristics. With this choice of slow control, the reduced fast subsystem becomes

$$\mathbf{e}' = \mathbf{0} \tag{21}$$

$$\xi' = \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{z}_r') - \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{0}) + A_\xi \xi + \mathbf{K}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r)\mathbf{u}_f \tag{22}$$

To stabilize the fast subsystem, the fast control $\mathbf{u}_f$ is designed as

$$\begin{bmatrix} \mathbf{G}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \\ \mathbf{K}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \end{bmatrix} \mathbf{u}_f = \begin{bmatrix} \mathbf{0} \\ \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{0}) - \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{z}_r') \end{bmatrix} \tag{23}$$

Thus, the composite control $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_f$ satisfies

$$\begin{bmatrix} \mathbf{G}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \\ \mathbf{K}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r) \end{bmatrix} \mathbf{u} = -\begin{bmatrix} \mathbf{F}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\dot{\mathbf{x}}_r) \\ \mathbf{L}(\mathbf{e},\xi,\mathbf{x}_r,\mathbf{z}_r,\mathbf{z}_r') \end{bmatrix} + \begin{bmatrix} A_e\mathbf{e} \\ A_\xi \xi \end{bmatrix} \tag{24}$$

assuming that the rank of $\begin{bmatrix} \mathbf{G}(.) \\ \mathbf{K}(.) \end{bmatrix} \geq (m+n)$.

The complete closed-loop and reduced slow subsystem for this control law are given as

$$\dot{\mathbf{e}} = A_e\mathbf{e} \tag{25}$$

$$\varepsilon\dot{\xi} = A_\xi \xi. \tag{26}$$

and

$$\dot{\mathbf{e}} = A_e\mathbf{e} \tag{27}$$

$$\mathbf{0} = A_\xi \xi. \tag{28}$$

respectively. Observe that with the proposed control law the nonlinear algebraic set of equations (Eq.17) have been transformed to a linear set of equations (Eq.28). With the proper choice of A_ξ, it is guaranteed that $\xi = \mathbf{0}$ is the unique manifold for both the complete and the reduced slow subsystems. Furthermore, this manifold is exponentially stable as can be deduced from the reduced fast subsystem

$$\mathbf{e}' = \mathbf{0} \tag{29}$$

$$\xi' = A_\xi \xi \tag{30}$$

Remark 1 The control law proposed in Eq.24 is independent of the perturbation parameter ε. Furthermore it is a function of $\mathbf{z}'_r$ that implies that the reference trajectory chosen for the fast states must be faster when compared to the reference of the slow states. Additionally, as for all singular perturbation techniques to work the closed-loop eigenvalues A_e and A_ξ must be chosen so as to maintain the time-scale separation.

3.2 Stability Analysis

Complete system stability is analyzed using the composite Lyapunov function approach[10]. Suppose that there exist positive definite Lyapunov functions $V(t,\mathbf{e}) = \mathbf{e}^T\mathbf{e}$ and $W(t,\xi) = \xi^T\xi$ for the reduced subsystems, with continuous first-order derivatives satisfying the following properties:

1. $V(t,\mathbf{0}) = 0$ and $\gamma_1||\mathbf{e}||^2 \leq V(t,\mathbf{e}) \leq \gamma_2||\mathbf{e}||^2\ \forall t \in \mathbb{R}^+, \mathbf{e} \in \mathbb{R}^m, \gamma_1 = \gamma_2 = 1,$
2. $(\nabla_{\mathbf{e}}V(t,\mathbf{e}))^T A_{\mathbf{e}}\mathbf{e} \leq -\alpha_1\mathbf{e}^T\mathbf{e}, \quad \alpha_1 = 2|\lambda_{\min}(A_{\mathbf{e}})|,$
3. $W(t,\mathbf{0}) = 0$ and $\gamma_3||\xi||^2 \leq W(t,\xi) \leq \gamma_4||\xi||^2\ \forall t \in \mathbb{R}^+, \xi \in \mathbb{R}^n, \gamma_3 = \gamma_4 = 1,$
4. $(\nabla_\xi W(t,\xi))^T A_\xi\xi \leq -\alpha_2\xi^T\xi, \quad \alpha_2 = 2|\lambda_{\min}(A_\xi)|.$

Next, consider the composite Lyapunov function $\nu(t,\mathbf{e},\xi) : \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^+$ defined by the weighted sum of $V(t,\mathbf{e})$ and $W(t,\xi)$ for the complete closed-loop system

$$\nu(t,\mathbf{e},\xi) = (1-d)V(t,\mathbf{e}) + dW(t,\xi), \quad 0 < d < 1 \tag{31}$$

The derivative of $\nu(t,\mathbf{e},\xi)$ along the closed-loop trajectories Eqs.25,26 is given by

$$\dot{\nu} = (1-d)(\nabla_{\mathbf{e}}V)^T\dot{\mathbf{e}} + d(\nabla_\xi W)^T\dot{\xi} \tag{32}$$

$$\dot{\nu} = (1-d)(\nabla_{\mathbf{e}}V)^T A_{\mathbf{e}}\mathbf{e} + \frac{d}{\varepsilon}(\nabla_\xi W)^T A_\xi\xi \tag{33}$$

$$\dot{\nu} \leq -(1-d)\alpha_1\mathbf{e}^T\mathbf{e} - \frac{d}{\varepsilon}\alpha_2\xi^T\xi \tag{34}$$

$$\dot{\nu} \leq -\begin{bmatrix}\mathbf{e}\\ \xi\end{bmatrix}^T \begin{bmatrix}(1-d)\alpha_1 & 0\\ 0 & \frac{d}{\varepsilon}\alpha_2\end{bmatrix} \begin{bmatrix}\mathbf{e}\\ \xi\end{bmatrix} \tag{35}$$

Following the approach proposed in Reference[3], add and subtract $2\alpha\nu(t,\mathbf{e},\xi)$ to Eq.35 to get

$$\dot{\nu} \leq -\begin{bmatrix}\mathbf{e}\\ \xi\end{bmatrix}^T \begin{bmatrix}(1-d)\alpha_1 & 0\\ 0 & \frac{d}{\varepsilon}\alpha_2\end{bmatrix} \begin{bmatrix}\mathbf{e}\\ \xi\end{bmatrix} + 2\alpha(1-d)V + 2\alpha dW - 2\alpha\nu \tag{36}$$

where $\alpha > 0$. Substitute in Eq.36 for the Lyapunov functions $V(t,\mathbf{e})$ and $W(t,\xi)$ to get

$$\dot{v} \leq -\begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix}^T \begin{bmatrix} (1-d)\alpha_1 - 2\alpha(1-d) & 0 \\ 0 & \frac{d}{\varepsilon}\alpha_2 - 2\alpha d \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} - 2\alpha v \tag{37}$$

If ε satisfies

$$\varepsilon < \varepsilon^* = \frac{\alpha_2}{2\alpha} \tag{38}$$

and provided $\alpha_1 > 2\alpha$, then from the definitions of α_2, α, and d it can be concluded that the matrix in Eq.37 is positive definite. Then the derivative of the Lyapunov function is lower-bounded by

$$\dot{v} \leq -2\alpha v \tag{39}$$

Since the composite Lyapunov function lies within the following bounds

$$(1-d)\gamma_1 ||\mathbf{e}||^2 + d\gamma_3 ||\xi||^2 \leq v(t,\mathbf{e},\xi) \leq (1-d)\gamma_2 ||\mathbf{e}||^2 + d\gamma_4 ||\xi||^2 \tag{40}$$

or,

$$\gamma_{11} \left|\left| \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} \right|\right|^2 \leq v(t,\mathbf{e},\xi) \leq \gamma_{22} \left|\left| \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} \right|\right|^2 \tag{41}$$

where $\gamma_{11} = \min((1-d)\gamma_1, d\gamma_3)$ and $\gamma_{22} = \min((1-d)\gamma_2, d\gamma_4)$, the derivative of the Lyapunov function can be expressed as

$$\dot{v} \leq -2\alpha\gamma_{11} \left|\left| \begin{bmatrix} \mathbf{e} \\ \xi \end{bmatrix} \right|\right|^2 \tag{42}$$

From the definition of the constants γ_{11}, γ_{22}, and α, and invoking Lyapunov's Direct Method[9], *uniform exponential stability in the large of* $(\mathbf{e} = \mathbf{0}, \xi = \mathbf{0})$ *can be concluded*. Furthermore, since the reference trajectory $\mathbf{x}_r(t)$ and $\mathbf{z}_r(t)$ is bounded, it is concluded that the states $\mathbf{x}(t) \to \mathbf{x}_r(t)$ and $\mathbf{z}(t) \to \mathbf{z}_r(t)$ as $t \to \infty$. Since the matrix $\begin{bmatrix} \mathbf{G}(.) \\ \mathbf{K}(.) \end{bmatrix}$ is restricted to be full rank, examining the expression for $\mathbf{u}$ in Eq.24 it is concluded that $\mathbf{u} \in \mathcal{L}_\infty$.

Remark 2 Recall that for the special case of state regulation the system dynamics in Eqs.14,15 become autonomous. In such a case, the result of global exponential stability is obtained with less-restrictive conditions on the Lyapunov functions $V(\mathbf{e})$, $W(\xi)$, and consequently $v(\mathbf{e},\xi)$. Similar conclusions were made in Reference[3] for the stabilization problem of a special class of singularly perturbed systems where the control affects only the fast states. Note that for the special class of systems considered in Reference[3], the non-diagonal elements of the matrix in Eq.37 are nonzero, and the bound on the parameter ε is slightly different.

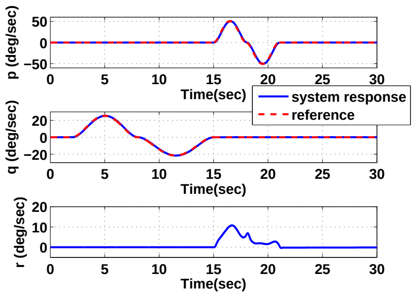
Remark 3 From Eq.37, a conservative upper bound for α is $\alpha < \frac{\alpha_1}{2}$, and consequently $\varepsilon^* \approx \frac{\alpha_2}{\alpha_1}$. Therefore, qualitatively this upper bound is indirectly dependent upon the choice of the closed-loop eigenvalues.

4 Numerical Simulation

The purpose of the example is to demonstrate the methodology and controller performance for an under-actuated, nonlinear, singularly perturbed system. The system studied is a nonlinear, coupled, six degree-of-freedom F/A-18A Hornet aircraft[5]. The combined longitudinal-lateral/directional maneuver requires tracking of the fast variables, in this case body-axis pitch and roll rates, while maintaining zero sideslip angle. Closed-loop characteristics such as stability, accuracy, speed of response and robustness are qualitatively analyzed. The maneuver consists of an aggressive vertical climb with a pitch rate of 25 deg/sec, followed by a roll at a rate of 50 deg/sec, while maintaining zero sideslip angle. The Mach number and angle-of-attack are assumed to be input-to-state stable. The initial conditions are a Mach number of 0.4 at $15,000$ feet, an angle-of-attack of 10 deg, and elevon angle of -11.85 deg. All other states are zero. The F/A-18A Hornet model is expressed in stability axes. Since it is difficult to cast the nonlinear aircraft model into the singular perturbation form of Eq.1-2 , the perturbation parameter ε is introduced in front of those state variables that have the fastest dynamics. This is done so that the results obtained for $\varepsilon = 0$ will closely approximate the complete system behaviour (with $\varepsilon = 1$). This is called the forced perturbation technique, and is commonly used in the aircraft literature [2, 12]. Motivated by experience and previous results, the six slow states are Mach number M, angle-of-attack α, sideslip angle β and the three kinematic states: bank angle ϕ, pitch-attitude angle θ, and heading angle ψ. The three body-axis angular rates (p,q,r) constitute the fast states. The control variables for this model are elevon δ_e, aileron δ_a, and rudder δ_r and are assumed to have sufficiently fast enough actuator dynamics. The convention used is that a positive deflection generates a negative moment. The throttle η is maintained constant at 80%, because slow engine dynamics require introduction of an additional time-scale in the analysis; this is a consideration which is beyond the scope of this paper. The aerodynamic stability and control derivatives are represented as nonlinear analytical functions of aerodynamic angles and control surface deflections. Quaternions are used to represent the kinematic relationships from which the Euler angles are extracted. The details of these relationships are discussed in Reference[15].

Results and Discussion

Simulation results in Figures 1-6 show that all controlled states closely track their references. At two seconds the aircraft is commanded to perform a vertical climb, and after eight seconds the pitch rate command changes direction and Mach number drops. The lateral/directional states and controls are identically zero until the roll command is introduced at time equals 15 seconds. Observe that all of the states asymptotically track the reference. Figure 2 shows that the elevon deflection remains within specified limits[5] throughout the vertical climb, and the commanded roll produces a sideslip angle which is negated by application of the rudder. The aileron and the rudder deflections remain within bounds while the aircraft rolls and comes back to level flight. The maximum pitch-attitude angle is 81 deg, maximum bank angle is 81 deg (Figure 4), and the maximum sideslip error is ± 4deg. The quaternions and the complete trajectory are shown in Figures 5 and 6 respectively. From Figure 6, note that after completing the combined climb and roll maneuver, the aircraft is commanded to remain at zero sideslip angle, roll rate, and pitch rate. It then enters a steady dive with all other aircraft states bounded. The controller response is judged to be essentially independent of the reference trajectory designed. The robustness properties of the controller are quantified by the upper bound ε^*. For this example, the design variables are $d = 0.5$, $\alpha_1 = 10$, $\alpha = 2$, and $\alpha_2 = 15$, so the upper bound becomes $\varepsilon^* = 7.5$. Therefore for all $\varepsilon < \varepsilon^*$ global asymptotic tracking is guaranteed and in this case $\varepsilon = 1$.



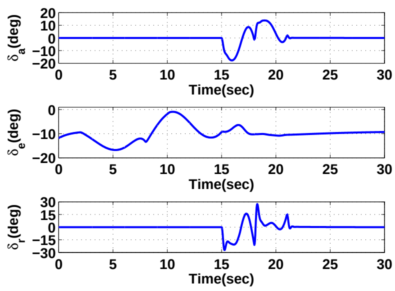
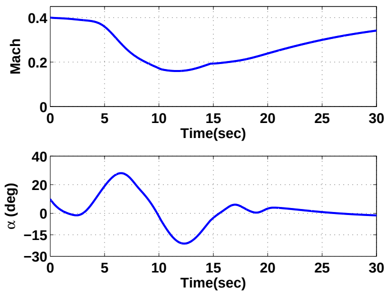


Fig. 1 Body-Axis Angular Rates

Fig. 2 Control Surface Deflections



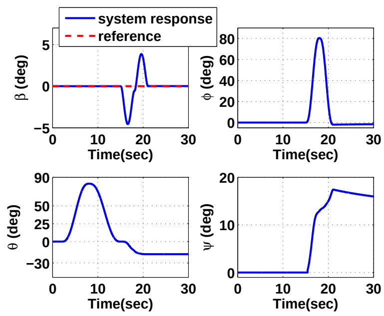


Fig. 3 Mach Number and Angle-of-Attack

Fig. 4 Sideslip Angle and Kinematic Angles

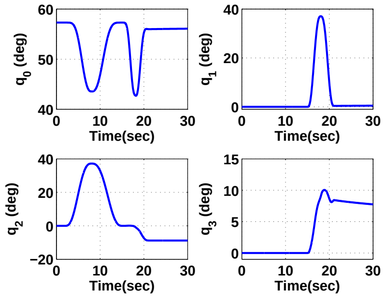


Fig. 5 Quaternion Parameters

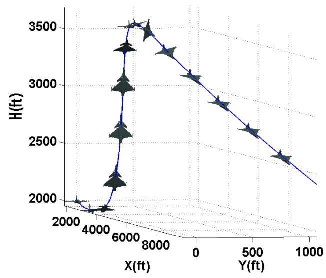


Fig. 6 Three-Dimensional Trajectory

5 Conclusions

A control law for global asymptotic tracking of both the slow and the fast states for a general class of nonlinear singularly perturbed systems was developed. A composite control approach was adopted to satisfy two objectives. First, it enforces the specified reference for the fast states to be 'the unique manifold' of the fast dynamics for all time. Second, it ensures that the slow states are tracked simultaneously as desired. Stability of the closed-loop signals was analyzed using the composite Lyapunov approach, and controller performance was demonstrated through numerical simulation of a nonlinear, coupled, six degree-of-freedom model of an F/A-18A Hornet. The control laws were implemented without making any assumptions about the nonlinearity of the six degree-of-freedom aircraft model. Based on the results presented in the paper, the following conclusions are drawn. First, both positive and negative angular rate commands were seen to be perfectly tracked by the controller and consistent tracking was guaranteed independent of the desired reference trajectory. Second, throughout the maneuver the controller demonstrated global asymptotic tracking even though the desired reference trajectory requires the aircraft to switch between linear and nonlinear regimes. This robust performance of the controller was shown to hold for all $\varepsilon < \varepsilon^* = 7.5$. Third, all closed-loop signals were bounded and the control surface deflections computed were smooth and within specified limits. Fourth, this technique does not require the knowledge of the perturbation parameter ε. This is an important consideration for systems such as aircraft, where quantifying this parameter can be difficult.

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