Kinetic State Tracking For A Class of Singularly Perturbed Systems

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Overview

- Problem Definition
  - Non-Standard Singularly Perturbed Model

- Geometric Singular Perturbation Theory

- Mathematical development & stability analysis

- Numerical Simulations
  - Generic two-degree-of-freedom nonlinear model
  - Nonlinear six degree-of-freedom simulation of an F/A-18A Hornet

- Conclusions & Future Work
Research Objective

- Nonlinear tracking control structures for:
  - **Two-time scale systems/ Singularly Perturbed Systems**
    - Examples: mechanical oscillators, airplanes, flexible robot link manipulators, ...
  - **Mathematical form:**
    \[
    \begin{align*}
    \dot{x} &= f(x, z) + g(x, z)u \\
    \epsilon \dot{z} &= l(x, z) + k(x, z)u \\
    y &= x
    \end{align*}
    \]

  - $x$ is the vector of slow variables,
  - $z$ is the vector of fast variables,
  - $\epsilon$ is a small positive parameter that captures the time scale property,
  - $y$ is the vector of outputs.
Problems Studied in Literature

- **‘Standard Singular Perturbation Model’**
  - Kokotovic (1986), Christofides (1996), …

\[ \begin{align*}
  \dot{x} & = f(x, z) + g(x, z)u \\
  \epsilon z & = l(x, z) + k(x, z)u \\
  y & = x \\
  \epsilon & = 0
\end{align*} \]

  - Assume that the algebraic equation has distinct real roots for the fast states

- **‘Non-Standard Singular Perturbation Model’**
  - Fridman (2001): Optimal control for regulation of the states
Challenge: Non-Standard Singularly Perturbed Model

- **Key Issues:**
  - The system is numerically stiff
  - Control instability issues
  - Transcendental equations may have multiple roots for the fast states

- **Approach:**
  - Model-reduction via Geometric Singular Perturbation Theory
  - Approximate solution of algebraic equations via centre manifold theorem
  - Composite control design

- **Benefits:**
  - Relieves the numerical stiffness
  - Controller computed in accordance of the speed of the states
  - Does not assume existence of isolated real roots
Geometric Singular Perturbation Theory
Model Reduction Technique
Geometric Singular Perturbation Theory

(Fenichel, 1979)

Derivatives wrt slow time scale $t$

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -x_2 \\
\dot{\epsilon}z &= -z
\end{align*}
\]

Derivatives wrt fast time-scale $\tau$

\[
\tau = \frac{t - t_0}{\epsilon}
\]

\[
\begin{align*}
\dot{x}_1 &= -\epsilon x_1 \\
\dot{x}_2 &= -\epsilon x_2 \\
\dot{z} &= -z
\end{align*}
\]

Develop reduced-order models. Substitute $\epsilon = 0$

\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -x_2 \\
\dot{z} &= 0
\end{align*}
\]

Slow Subsystem

Fast Subsystem
Geometric Singular Perturbation Theory

- Reduced-order models approximate the behaviour of the complete system.

**Complete System**

\[
\begin{align*}
  x_1 &= -x_1 \\
  x_2 &= -x_2 \\
  \epsilon z &= -z
\end{align*}
\]

- There exists a smooth manifold \( \mathcal{M}_0 : z = 0 \) of dimension 2.

**Slow Subsystem**

\[
\begin{align*}
  x_1 &= -x_1 \\
  x_2 &= -x_2 \\
  z &= 0
\end{align*}
\]

**Fast Subsystem**

\[
\begin{align*}
  x_1' &= 0 \\
  x_2' &= 0 \\
  z' &= -z
\end{align*}
\]
Handling Tracking Requirements

- For the complete system
  \[ \dot{x} = f(x, z) + g(x, z)u \]
  \[ \varepsilon z = l(x, z) + k(x, z)u \]
  \[ y = x \]

- Generate reduced order models

  **Slow Subsystem**
  \[ \dot{x} = f(x, z) + g(x, z)u \]
  \[ 0 = l(x, z) + k(x, z)u \]
  \[ y = x \]

  **Fast Subsystem**
  \[ x' = 0 \]
  \[ z' = l(x, z) + k(x, z)u \]
  \[ y = x \]

- Find the manifold \( M_0 : z = Z_0(x, u) \) such that it satisfies
  \[ l(x, z) + k(x, z)u = 0 \]

- Design tracking controller for the reduced system
- Design controller to stabilize the fast states about the manifold
Centre Manifold

- Linearize the system about \((\epsilon = 0, x, Z_0(x, u))\)
  \[ x' = \epsilon (f(x, z) + g(x, z)u) \]
  \[ z' = l(x, z) + k(x, z)u \]
  \[ \epsilon' = 0 \]

- Linearized system:
  \[ \Delta x' = F \Delta x + F_1 \Delta z + G \Delta u \]
  \[ \Delta z' = L \Delta z + L_1 \Delta x + K \Delta u \]

- If all the eigenvalues of \(F\) have zero real parts while \(L\) has negative real parts, then \(M_0\) is the centre manifold
Centre Manifold Theorem

(Carr, 1981)

- **Conditions** on the system:
  - Origin is the stable equilibrium,
  - System vector fields are sufficiently smooth,
  - System has a centre manifold,

- **Statement:**
  For small initial conditions, then the manifold can be approximated to any degree of accuracy.

- **Example:**
  \[ x = xz + ax^3 + bz^2x \]
  \[ \epsilon z = -z + cx^2 + dx^2z \]
  - Algebraic equation: \(-z + cx^2 + dx^2z = 0\)
  - Approximate manifold \( \Phi_0 = cx^2 \ | x | < 1 \)
Mathematical Formulation of the Control Law

- Complete Model:
  \[ \dot{x} = f(x, z) + g(x, z)u \]
  \[ \dot{\epsilon z} = l(x, z) + k(x, z)u \]
  \[ y = x \]

- Tracking error: \[ e = x - x_r \]
- Error between fast states and approximate manifold: \[ \zeta = z - \Phi_0(e, u) \]
- Transform the equilibrium to origin:

\[
\begin{align*}
\dot{e} &= f^*(e, \zeta + \Phi_0) + g^*(e, \zeta + \Phi_0)u; \\
\dot{\epsilon \zeta} &= l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)u - \epsilon \Phi_0
\end{align*}
\]
Composite Control Design

\[ \dot{e} = f^*(e, \zeta + \Phi_0) + g^*(e, \zeta + \Phi_0)u; \]

\[ \epsilon \dot{\zeta} = l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)u - \epsilon \Phi_0 \]

- **Slow Subsystem**
  \[ \dot{e} = f^*(e, \Phi_0(e, u_s)) + g^*(e, \Phi_0(e, u_s))u_s \]

- **Control Law**
  \[ u_s = -\left( g(e) \right)^{-1} \left( K_e e + \tilde{f}(e) \right) \]

- **Fast Subsystem**
  \[ \dot{e}' = 0; \]

\[ \zeta' = l^*(e, \zeta + \Phi_0) + k^*(e, \zeta + \Phi_0)(u_s + u_f) \]

- **Control Law**
  \[ u_f = -\left( k(e) \right)^{-1} \left( K_\zeta \zeta + \tilde{l}(e) \right) \]

Control applied to the complete system (independent of \( \epsilon \))

\[ u = u_s(e) + u_f(e, \zeta) \]
Lyapunov Analysis

Lyapunov Function Candidate:

\[ \nu(e, \zeta) = e^T e + \zeta^T \zeta \]

Time derivative about closed-loop dynamics:

\[ \dot{\nu} = - \left[ \|e\| \right]^T \begin{bmatrix} 2K_e & -\beta_1 \frac{\beta_1}{2} \\ -\beta_1 \frac{\beta_1}{2} & 2 \frac{K_\zeta}{\epsilon} \end{bmatrix} \left[ \|e\| \right] - \|\zeta\| \|2\Phi_0\| \]

For \( \epsilon < \frac{8K e K_\zeta}{\beta_1^2} \) stability of the system is guaranteed.

Note: If the exact solution of manifold is computed, asymptotic stability can be proved.
Example #1

Generic two degree-of-freedom model:

\[ \dot{x} = -x + (x + \kappa - \lambda)z + u; \]
\[ \epsilon \dot{z} = x - (x + \kappa)z + z^2 + u \]

Given: \( \epsilon = 0.2, \lambda = 0.5, \kappa = 1 \)

Approximate manifold: \( \Phi_0 = \frac{x + u}{x + \kappa} \)

Test Cases:
- Tracking performance demonstration for nonlinear model.
- Tracking performance demonstration without the nonlinearity in the fast dynamics.
Reference Trajectory Tracking

Time History of the Slow States

With Nonlinear Terms

Without Nonlinear Terms
Time History of the Fast States

With Nonlinear Terms

Without Nonlinear Terms
Time History of Control

With Nonlinear Terms

Without Nonlinear Terms
Example #2


- Slow states: \( x = [M, \alpha, \beta, \phi, \theta, \psi]^T \)

- Fast states: \( z = [p, q, r]^T \)

- Control Variables: \( u = [\eta, \delta_e, \delta_a, \delta_r]^T \)

- Objective: Turning maneuver to be executed at zero sideslip and specified angle-of-attack profile.
Response of the Slow States

- Mach
- $\alpha$ (deg)
- $\beta$ (deg)
- $\phi$ (deg)
- $\psi$ (deg)
Response of the Angular Rates & Control
Conclusions

- Developed trajectory-following controllers for non-standard singularly perturbed system using geometric singular perturbation theory as a model-reduction technique.

- Using concepts from centre-manifold theory, approximate solutions of the nonlinear algebraic equations are obtained.

- Controllers are independent of the perturbation parameter, $\varepsilon$.

- Stability of the complete nonlinear system is shown for $\varepsilon \leq \varepsilon^*$.
Future Research Extensions

- The fast states are restricted by the slow states
  - For $x \rightarrow x_r$, it is required that $z \rightarrow \Phi_0(x)$
  - Extensions for fast state tracking are required
  - Example: Propulsion Controlled Aircraft

- Current research assumes that the control appears linearly
  \[
  \dot{e} = f^\ast(e, \Phi_0(e, u_s)) + g^\ast(e, \Phi_0(e, u_s))u_s
  \]
  \[
  u_s = -\left(\tilde{g}(e)\right)^{-1}\left(K_e e + \tilde{f}(e)\right)
  \]
  - Extensions for nonlinear-in-control systems is required.

- Autonomous control of Highly Reconfigurable Structures
  - Adaptive-Reinforcement Learning (Valasek et.al, 2004)
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QUESTIONS?