PRELIMINARY RESULTS OF VEHICLE FORMATION CONTROL USING DYNAMIC INVERSION

Rafael E. Caicedo*, John Valasek†, and John L. Junkins‡
Texas A&M University, College Station, TX 77843-3141

ABSTRACT
Results are presented from a preliminary study to determine the potential benefits of utilizing dynamic inversion inspired by a structural analogy for formation control of a system of vehicles. Vehicles are modeled as unconstrained mass particles under the influence of control forces in an inertial reference frame. For a system of particles restricted to motion in one dimension only, equations of motion are developed, and a corresponding dynamic inversion control law is proposed. A dissipative mechanical structure analogy is used to establish a family of feedback controllers. The conditions for global asymptotic stability are presented, and information requirements are discussed. It is shown that a stable system can be generated with each particle obtaining measurements for the mass center location of the system, and the position of the particles immediately next to it. A simulation of the response of a system to initial errors and desired trajectories using the proposed control law is presented yielding satisfactory results.

INTRODUCTION
Formation control, in recent years, has been a topic of significant interest to researchers. Of particular interest is the control of cooperative teams of vehicles to perform tasks such as reconnaissance missions and cooperative strikes. More specifically, the advantages of formation flight of air vehicles, manned or unmanned, include the increased capability of performing missions that would otherwise be more difficult for a single, larger aircraft. Additionally, it is known that formation flight yields a reduction in the total drag of a wing, or follower, aircraft trailing behind a lead aircraft. A significant amount of work has been devoted to designing control systems for aircraft in formation. In particular, Pachter et al.¹ have developed a linear controller for formation flight, while Singh et al.² have developed an adaptive feedback linearizing controller. More recently, the work of Olalfi-Saber and Murray³ provides a theoretical framework for stability analysis and distributed control of multi-agent formations in two or three dimensions. Giulietti et al.⁴ developed a controller in which the flight vehicles retain their distance with respect to an imaginary point in the formation.

Most of the controllers described above rely on the existence of a leader, whose actions are followed by other vehicles. Instead of using a leader or an imaginary point to guide the formation, this paper explores the advantages of using the mass center of the system, in essence a virtual formation, as the guide. The case of one-dimensional formation control is investigated, equations of motion are developed using a Lagrangian approach, and a corresponding dynamic inversion control law is generated. Lastly, stability characteristics and information requirements are analyzed.

ONE-DIMENSIONAL FORMATION CONTROL

This section presents the application of the proposed approach to a system of particles constrained to move in one dimension only. Equations of motion for the system are developed, and the stability of the formulation is studied.

Equations of Motion
As with any dynamical system, the equations describing the dynamics of the system need to be developed. First, a coordinate system is defined as presented in figure 1.

In figure 1, there exist n particles that are collocated in one-dimensional space, and can only move in a straight line. This system could be representative of a line of vehicles following each other. The mass of each particle is denoted as mᵢ, the inertial position of each particle as xᵢ, and the control force acting on each particle as uᵢ. The position state of the system can be described with the...
vector \( \mathbf{x} \) and the control forces with the vector \( \mathbf{u} \) as follows:

\[
\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}^T,
\]

\[
\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_n \end{bmatrix}^T.
\]

At this time, a new set of generalized coordinates is chosen to describe the state of the system. The choice of these generalized coordinates relies on the intuitive notion that the state of a formation system is described by two key parameters: the location of the formation, and the geometry of the formation. The generalized coordinates for the system are chosen to be the mass center of the system, which is a relative measure of where the formation is, and the distances between the particles in the formation, a relative measure of the current geometry of the system. Therefore, the generalized coordinates are defined as follows:

\[
x_c = \frac{1}{m} \sum_{i=1}^{n} m_i x_i \quad \text{where} \quad m = \sum_{i=1}^{n} m_i
\]

and

\[
x_{k,k+1} = x_{k+1} - x_k.
\]

The coordinate transformation described above yields the new position state vector, \( \mathbf{q} \), as follows:

\[
\mathbf{q} = \begin{bmatrix} x_c & x_{12} & x_{3} & \cdots & x_{n-1,n} \end{bmatrix}^T,
\]

which is equivalent to the following transformation:

\[
\mathbf{q} = \mathbf{R}^{-1} \mathbf{x} \quad (1).
\]

A Lagrangian approach is utilized to develop the equations of motion for this system of particles. The equations of motion of the system can be generated using LaGrange's equations:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q} \quad (2)
\]

In equation 2, \( T \) represents the kinetic energy of the system, \( \mathbf{q} \) represents the generalized coordinate vector defined previously, and \( \mathbf{Q} \) represents the generalized forces acting on the system. It is required to obtain both \( T \) and \( \mathbf{Q} \) to obtain the equations of motion.

First, the kinetic energy of the system is determined, and is given by:

\[
T = \frac{1}{2} \mathbf{x}^T \mathbf{M} \dot{\mathbf{x}},
\]

substituting equation 1 into the above expression, the following expression is obtained:

\[
T = \frac{1}{2} \mathbf{q}^T \mathbf{R}^T \mathbf{M} \mathbf{R} \mathbf{q}.
\]

Furthermore, a matrix \( \mathbf{\tilde{M}} \) is defined as follows:

\[
\mathbf{\tilde{M}} = \mathbf{R}^T \mathbf{M} \mathbf{R}.
\]

Because the matrix \( \mathbf{\tilde{M}} \) is formed from a diagonal matrix whose values are positive, \( \mathbf{\tilde{M}} \) is not only symmetric but also positive definite. It has the following symmetric form:

\[
\mathbf{\tilde{M}} = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 \\
0 & m_{22} & m_{23} & \tilde{M}_{24} & \\
0 & \tilde{M}_{23} & \tilde{M}_{33} & \cdots & \tilde{M}_{3,n} \\
0 & \tilde{M}_{24} & \cdots & \tilde{M}_{n-1,n} & \\
0 & \tilde{M}_{n-1,n} & \cdots & \tilde{M}_{n,n} & 
\end{bmatrix}
\]

Finally, the kinetic energy expression becomes:

\[
T = \frac{1}{2} \mathbf{q}^T \mathbf{\tilde{M}} \mathbf{q} \quad (3).
\]

An expression for the generalized force vector, \( \mathbf{Q} \), can be obtained easily using a virtual work approach. Consider the inertial coordinates \( x_i \), a force \( u_i \) acts through a displacement of the inertial coordinate \( \delta x_i \). As such, the virtual work done by the force vector \( \mathbf{u} \) is given by:

\[
\delta W = \sum_{i=1}^{n} u_i \delta x_i = \mathbf{u}^T \delta \mathbf{x}.
\]

Given the coordinate transformation between the inertial coordinates and the generalized coordinates, the virtual work can be expressed in terms of the generalized coordinates:
\[ \delta W = u^T \delta x = u^T \delta (Rq) = u^T R \delta q \] (4).

The virtual work of the system is also defined as the work done by the generalized forces through a displacement of the generalized coordinates as shown in equation 5:

\[ \delta W = Q^T \delta q \] (5)

Comparison of equations 4 and 5 reveals the generalized force vector:

\[ Q = R^T u \] (6).

Substituting equations 6 and 3 into equation 2 reveals the equations of motion for the system of particles:

\[ \ddot{\tilde{M}} q = R^T u \] (7).

**Dynamic Inversion Control Law**

With the given equations of motion presented in equation 7, a dynamic inversion control law can be developed. The objective of this control law will be to track a desired generalized coordinate input. As mentioned previously, the generalized coordinates are chosen as the parameters one generally wishes to control in a vehicle formation. In general, the desired position of the formation and geometry are known. The desired state of the generalized coordinates is denoted as \( q_{\text{des}} \) and with this vector, which is a function of time, a new error variable is defined as:

\[ z = q - q_{\text{des}} \].

This error definition yields an error function that is continuous and differentiable as long as both \( q \) and \( q_{\text{des}} \) are continuous and differentiable. Since \( q \) is a position-based set of generalized coordinates, it will be continuous in time because the position of a particle cannot change instantaneously. However, \( q_{\text{des}} \) is a designer-specified vector, so it is required that the desired generalized coordinate vector remains a continuous differentiable function of time. Differentiating the error vector \( z \) twice, the following expressions are obtained:

\[ \ddot{z} = \ddot{q} - \ddot{q}_{\text{des}} \]
\[ \ddot{q} = \dddot{z} + \dddot{q}_{\text{des}} \] (8).

Substituting equation 8 into equation 7 results in an expression for the dynamics of the errors given by the control forces \( u \) and desired trajectory vector \( q_{\text{des}} \):

\[ \dddot{M}(\ddot{z} + \ddot{q}_{\text{des}}) = R^T u \]
\[ \dddot{M} \dot{z} = R^T u - \dddot{M} \dot{q}_{\text{des}} \] (9).

Equation 9 permits the development of a dynamic inversion control law. A control law of the following form is proposed:

\[ u = (R^T)^{-1} (\dddot{M} \dot{q}_{\text{des}} - \dddot{C} \dot{z} - \dddot{K} \dot{z}) \] (10).

In equation 10, the matrices \( \dddot{C} \) and \( \dddot{K} \) are matrices of gains selected by the designer. Their nature will be studied in the following sections. Substituting equation 10 into equation 9 and rearranging the terms results in the following error dynamics equation:

\[ \dddot{M} \dot{z} + \dddot{C} \dot{z} + \dddot{K} \dot{z} = 0 \] (11).

Using Lyapunov stability theory and the information requirements of the system, the nature of the designer-selected parameters of the system will be determined.

**Lyapunov Stability Analysis**

According to Lyapunov stability theory, a system whose dynamics are expressed in the form:

\[ \dot{x} = f(x) \] (12),

has an equilibrium that is globally asymptotically stable if there exists a Lyapunov function, \( V(x) \), such that:

- \( V(x) \) is positive definite
- \( \dot{V}(x) \) is negative definite
- \( V(x) \) approaches infinity as \( ||x|| \to \infty \).

Ideally it is desired for the system to be globally asymptotically stable. To express the dynamics of the system in the form presented by equation 12, the following state vector is defined:

\[ \bar{z} = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \]

and the closed-loop equations of motion in equation 11 are written as:

\[ \dot{\bar{z}} = \begin{bmatrix} 0 & 1 \\ -\dddot{M}^{-1} \dddot{K} & -\dddot{M}^{-1} \dddot{C} \end{bmatrix} \bar{z}, \]

where \( I \) is the identity matrix of size \( n \).

At this time, a Lyapunov function is created to study the nature of the \( \dddot{C} \) and \( \dddot{K} \) matrices. The objective is to determine the requirements of these matrices as to yield a globally asymptotically stable system. The proposed Lyapunov function is:

\[ V(\bar{z}) = \frac{1}{2} \bar{z}^T \begin{bmatrix} \dddot{K} & 0 \\ 0 & \dddot{M} \end{bmatrix} \bar{z} = \frac{1}{2} \dddot{z}^T \dddot{M} \dddot{z} + \frac{1}{2} \dddot{z}^T \dddot{K} \dddot{z} \]

For this Lyapunov function to be positive definite, \( \dddot{K} \) must be positive definite (recall that \( \dddot{M} \) is positive definite). The derivative of the Lyapunov function is given by:

\[ \dot{V}(\bar{z}) = \dddot{z}^T \dddot{M} \dddot{z} + \dddot{z}^T \dddot{K} \dddot{z} \]
\[ \dot{V}(\bar{z}) = -\dddot{z}^T (\dddot{C} \dddot{z} + \dddot{K} \dddot{z}) + \dddot{z}^T \dddot{K} \dddot{z} \]
\[ \dot{V}(\bar{z}) = -\dddot{z}^T \dddot{C} \dddot{z}, \]
which implies that $\hat{C}$ must be positive definite for stability. However, $\dot{V}(Z)$ can be zero if $\dot{z} = 0$ and $z$ is any arbitrary value. Junkins and Kim state that asymptotic stability can be shown if the first non-zero odd derivative in the set $Z$ where $\dot{V}(Z)$ vanishes is negative definite.\(^7\) The second derivative of $V(Z)$ computed to be:

$$\ddot{V}(Z) = -2\dot{z}^T \hat{C} \dot{z}$$

which is equal to zero in the domain $Z$ where $\dot{V}(Z)$ vanishes because $\dot{z} = 0$. Differentiating again yields:

$$\dddot{V}(Z) = -2\ddot{z}^T \hat{C} \ddot{z} - 2\dot{z}^T \hat{C} \dot{z}$$

eliminating the second term ($\dot{z} = 0$ in $Z$), and expanding the first term yields:

$$\ddot{V}(Z) = 2(-z^T \hat{C} \dddot{z} + \dddot{z}^T \hat{C} \dot{z} + \dddot{z}^T \hat{C} \ddot{z})$$

and, again noting that $\dot{z} = 0$ in $Z$, the above reduces to:

$$\ddot{V}(Z) = -2(z^T \hat{C} \ddot{z} + \dddot{z}^T \hat{C} \dot{z} + \dddot{z}^T \hat{C} \ddot{z})$$

which can be shown to be a negative definite function in the domain specified by $Z$: $V'(Z) = -2(s^T \hat{C}s)$, where $s = \hat{M}^{-1} \hat{K} \dot{z}$ and asymptotic stability is shown for the system.

Lastly, the behavior of $V(Z)$ as $\|Z\| \to \infty$ is to be analyzed. First, notice that $V(Z)$ can be expressed as:

$$V(Z) = \frac{1}{2} z^T A z \quad (13)$$

where $A$ is a symmetric matrix. The following transformation is implemented:

$$Z = Py$$

where $P$ is a matrix containing all the normalized eigenvectors $e_i$ of $A$ in the form $P = [e_1 \ e_2 \ \ldots \ e_{2n}]$. Substituting the transformation for $Z$ into equation 13 yields:

$$V(Z) = \frac{1}{2} y^T P^T A P y .$$

It can be shown that $P^{-1} = P^T$, and that the product of $P^T A P$ yields a diagonal matrix whose elements are the eigenvalues of $A$.\(^6\) Additionally, since $P^T P = I$, the order of $Z$ is the same as the order of $y$, indicating that as the norm of $y$ approaches infinity ($\|y\| \to \infty$), so does the norm of $Z$. Equation 13 can be rewritten as:

$$V(Z) = \frac{1}{2} y^T P^T A P y = \frac{1}{2} \sum_{i=1}^{2n} \lambda_i y_i^2$$

which indicates that $V(Z) \to \infty$ as $\|y\| \to \infty$, and since the order of $y$ is the same as the order of $Z$, $V(Z) \to \infty$ as $\|Z\| \to \infty$. This result completes the third and final requirement for global asymptotic stability of the system.

Information Requirements Analysis

An important aspect of cooperative control is the amount of information that must be transferred (and known) by the vehicles in the system. It is ideal for each vehicle to know the least amount of information possible and still be able to maintain the formation. This section will study the information requirements for the control law developed.

Again, note the form of the proposed control law:

$$u = (R^T)^{-1} (\hat{M}q_{des} - \hat{C}z - \hat{K}z),$$

additionally, note the form of $(R^T)^{-1}$:

$$\begin{bmatrix}
m_1 & -1 & 0 & 0 & \ldots & 0 \\
m_2 & 1 & -1 & 0 & \ldots & 0 \\
m_3 & 0 & 1 & -1 & \ldots & 0 \\
m_4 & 0 & 0 & 1 & \ldots & 0 \\
m_5 & \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\
m_n & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

Observing the control law above, it is seen that the only matrices over which the designer has control are the $\hat{K}$ and $\hat{C}$ matrices. Observing the products, $(R^T)^{-1} \hat{K}z$, and $(R^T)^{-1} \hat{C}z$, one can see that if both $\hat{K}$ and $\hat{C}$ are diagonal matrices, then the force applied to the $i_{th}$ particle due to these matrices will be a function of the location of the mass center of the system, along with the position of the particles next to the $i_{th}$ particle. For example, the 2\textsuperscript{nd} particle in the formation needs to measure or sense the distance to particles 1 and 3. For this reason, these matrices are chosen to be diagonal. If they were non-diagonal, they would couple the
measurements required to generate the control forces for each particle, requiring more information to be known by each particle, making the system more complex and more difficult to implement.

Lastly, note that the multiplication \((R^T)^{-1}\dot{M}\) does yield a fully populated matrix. This indicates that there is coupling between the desired coordinates of the system in the control law. However, this is not considered to be an impediment because it is assumed that the particles can have knowledge ahead of time of all the desired generalized coordinates for all particles. In other words, each particle must know, for all time, what all the measurements are supposed to be but it does not need to know the actual measurements at all times. As already stated, it only needs to know the current location of the mass center and the distance to the particles immediately next to it.

Of relevance is to clarify that there exists a difference between knowing where the system mass center is and knowing the position of each and every particle in the system. Having system mass center information does not necessarily mean that each particle must know the position of every other particle in the system. The former means that one agent, which could be a particle in the system or an external communications relay, provides each particle with the most current location of the system mass center as shown in figures 2, 3 and 4. The latter means that each particle must communicate with each and every one of the other particles in the system as shown in figure 5. Figures 2, 3, 4, and 5 illustrate the difference using an example with four particles.

In figure 2, one of the particles acts as a communications relay agent. It communicates with the other particles independently and separately. Each particle provides the agent with its current mass center location. The agent processes this information and provides each particle with the most up-to-date system mass center location. The number of communication links required for a system of particles using this approach is n-1 links, where n is the number of particles in the system. A disadvantage of this approach is that it may be limited by “how far back” communications are possible. For example, the communication between particle 2 and 4 may be difficult due to hardware constraints such as wireless modem range. Another consideration of this scenario is that if the relay agent is lost, the system must be able to reconfigure and designate a new relay agent.

Figure 3 presents a scenario where each particle communicates with the particles immediately adjacent to it. In this scenario, the system mass center information travels through the links, going back and forth between the beginning and end of the particle string. For example, particle 4 uses its last-known system mass center location along with its current mass center location to update the system mass center location. It relays the updated system mass center location to particle 3, and the process is repeated throughout the string. The number of communication links required for a system of particles using this approach is n-1 links, where n is the number of particles in the system. The advantage of this scenario is that communication is only required among adjacent particles, but the disadvantage is that there may exist a significant time delay in the information flow. That is, for example, it may take too long for particle 1 to receive the system mass center information with an updated particle 4 position.

Figure 4 presents a scenario in which an external agent (for example, a ground station) collects information and then transfers the system mass center location to each particle. It operates just the same way as the scenario in figure 2 does, but it requires n communication links for a system with n particles. Advantages of this approach include that there is no need to reconfigure the system if a particle is lost as there is in the scenario presented in figure 2. The disadvantage of this approach when compared to the scenarios in figure 2 and 3, aside from requiring more communication links, is that an external independent agent may not be available.
Summary: Control Law Implementation

Requirements

To summarize, the control law presented is globally asymptotically stable if the following conditions are met:

- The generalized coordinate vector, \( \mathbf{q}_{\text{des}} \), is continuous and differentiable in time.
- The \( \mathbf{K} \) and \( \mathbf{C} \) matrices are diagonal matrices, whose elements are all positive.
- Each particle has access to the values of all masses, all desired generalized coordinates for all time, measurements of the current distance to the particles immediately next to it, and the current location of the mass center.

**NUMERICAL EXAMPLE**

A numerical example was created to observe the response of a system as described in this paper to initial position errors. Using Simulink®, a system of five particles was simulated. The coordinate system used for this example was that shown in figure 1. Each particle had a mass of 1 mass unit, and was under the influence of a control force as shown in figure 1. The control law used was that shown in equation 10. The \( \mathbf{K} \) and \( \mathbf{C} \) matrices were chosen to be identity matrices.

The initial conditions of the simulation were given by inertial positions of \(-5, -4, -3, -2, \) and \(-1\) length units for particles 1 through 5, respectively. Their inertial velocities were initially zero. The particles were, initially, separated by an undesired distance of 1 length unit, and the formation mass center was incorrect at \(-3\) length units. The desired trajectory the particles were to follow was a sine wave of unit length amplitude and period 4\( \pi \) time units. Also, the desired spacing between the particles was 2 length units between two adjacent particles. In compliance with the requirements described in this paper, each particle had access to mass center location and distance to its immediate neighbors measurements.

| Figure 4. External Relay Agent Information Flow |
| Figure 5. All-Particle-Exchange Information Flow |
| Figure 6. System Response To Initial Errors and Desired Trajectories |

American Institute of Aeronautics and Astronautics
Figure 6 presents the time history of the inertial position of the particles. It is seen that the particles both position themselves in the appropriate locations with respect to each other and follow the sinusoidal wave accordingly. Figure 7 presents the generalized coordinate errors and, as expected, they all asymptotically approach zero.

ACKNOWLEDGEMENTS
This research is funded by the National Defense Science and Engineering Graduate Fellowship in conjunction with the Army Research Office. This support is gratefully acknowledged by the authors.

REFERENCES

CONCLUSIONS AND FURTHER WORK
This paper has presented the potential benefits of utilizing an approach based on a structural analogy to the formation control of vehicles constrained to move in one dimension. It was found that an asymptotically stable formation can be achieved if the system parameters meet certain requirements, and if each vehicle is provided with current mass center information, along with current position of the vehicles immediately next to it. An important result of this formulation is that each vehicle does not need to know the position of all the vehicles at all time, but rather their desired position trajectories.

Further research of this formation control approach will involve the study of unconstrained particles in two dimensions, followed by the study of non-holonomic vehicles capable of moving in two and three dimensions. For these studies, the objective will be to develop control laws that are similar to the one presented in this paper. The control laws are to be both asymptotically stable and requiring the least amount of information transfer possible. Also, optimization of the designer-selected matrices, string stability of the system, and collision avoidance algorithms are valid research areas to be explored.